Noname manuscript No.
(will be inserted by the editor)

The Value of Storage in Securing Reliability and Mitigating Risk in Energy Systems

Ali ParandehGheibi ·
Mardavij Roozbehani ·
Munther Dahleh ·
Asuman Ozdaglar

Received: date / Accepted: date

Abstract The paper examines the value of ramp-constrained storage in securing reliability and mitigating risk in energy systems with uncertain supply and demand, and friction in the main supply source. Reliability is defined as the expected discounted cost of energy deficits over an infinite horizon, whereas risk is defined as the probability of incurring a large energy deficit. The nominal supply and demand are assumed to match perfectly, while deviations from the nominal values are modeled as random shocks with stochastic arrivals. Due to friction, the random shocks cannot be tracked by the main supply sources. Storage, on the other hand, is assumed frictionless as a supply source and can be used to compensate for the energy deficit shocks, though it cannot be filled up instantaneously. The storage control problem is formulated as an optimal control problem with the objective of maximizing system reliability. It is shown that when the stage cost is linear in the size of the energy deficit, the optimal control policy is myopic in the sense that all deficit shocks will be compensated up to the available level of storage. However, when the stage cost is strictly convex, it may be optimal to accept a small energy deficit in the interest of maintaining a higher level of reserve, which can help avoiding a large energy deficit in the future. The value of storage capacity in improving

A. ParandehGheibi is a research affiliate at the Research Laboratory of Electronics (RLE) at the Massachusetts Institute of Technology (MIT), Cambridge, MA 02139.
E-mail: parandeh@mit.edu

M. Roozbehani is a Principal Research Scientist at the Laboratory for Information and Decision Systems (LIDS) at MIT.
E-mail: mardavij@mit.edu

M. Dahleh is a professor of Electrical Engineering and Computer Science (EECS) at MIT's Laboratory for Information and Decision Systems (LIDS)
E-mail: dahleh@mit.edu

A. Ozdaglar is a professor of Electrical Engineering and Computer Science (EECS) at MIT's Laboratory for Information and Decision Systems (LIDS)
E-mail: asuman@mit.edu

reliability, as well as the effects of the associated optimal policies under different stage costs on risk, i.e., the tail distribution of large energy deficits are examined.

Keywords Storage, Ramp Constraints, Reliability, Probability of Large Energy Deficits

1 Introduction

Supply and demand in energy systems are subject to exogenous and unpredictable shocks due to failure of generators or transmission equipment, or unexpected changes in weather conditions. In addition, energy systems in the United States and around the globe are moving towards integration of distributed and renewable energy resources at a large scale. Due to the volatile and intermittent nature of these resources, this trend is likely to increase the magnitude and frequency of impulsive supply shocks in energy networks. We are interested in understanding the value of storage capacity in improving the reliability of a system with volatile supply and demand, the limitations that cannot be overcome by additional storage capacity due to physical ramp constraints, and finally, the impacts of different control and pricing policies on system reliability and risk.

Formally, we define reliability as the expected infinite-horizon discounted cost of energy deficits. We will also refer to this reliability metric as the cost of energy deficit (CED) metric. The reliability value of storage is in turn defined as the maximal (i.e., under the optimal policy) relative improvement in system reliability for a given storage capacity. Another metric of interest in this paper is risk, defined as the probability of incurring an uncompensated energy deficit larger than a certain threshold[1]. We derive the associated optimal policies and characterize the improvement in system reliability as a function of storage capacity, and then qualitatively examine the effects of different policies on risk in the system.

We consider an abstract model of a system consisting of renewable generation, conventional generation, and storage. The system is subject to random arrivals of energy deficit shocks, due to defaults by the renewable generation or unexpected surges in the demand. It is assumed that the conventional generator is relatively slow and cannot ramp up fast enough to cover for the energy deficit shocks. The storage has finite capacity and a ramp constraint on charging, but no constraint on discharging. Thus, it may be used to partially or completely mask the shocks to avoid energy deficits or supply deficits. Note that this specific model of storage approximates very well the charge and discharge characteristics of various electricity storage technologies [1] including

[1] This metric is closely related to the popular notion of Loss of Load Probability (LOLP) [19] widely used in the energy systems literature. LOLP generally refers to the probability that supply will be insufficient to meet the demand. We adopt the notion of *risk* to emphasize the events that supply deficits would be relatively large.

pumped hydro, certain battery technologies, and the grid inertia. The key feature of this approximate model is that a significant amount of energy can be extracted in a short period of time, while it takes time to restore the consumed energy. Even though our model is relatively abstract simplification of the real energy systems, it helps gain insight on the effect of different control policies on the reliability of energy. Moreover, its simplicity allows us to provide some analytical characterization of the optimal control policies for operating energy storage systems together with volatile energy sources.

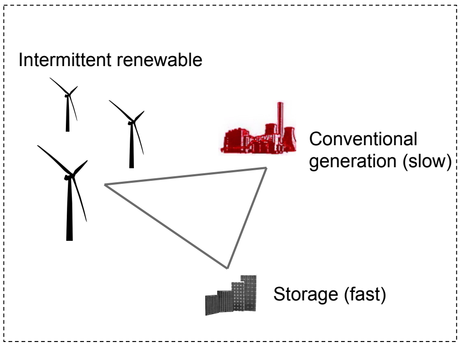

Fig. 1 A virtual power plant consisting of energy storage, renewable generation, and conventional generation.

This model can be interpreted in at least two different ways. The first is the model of a *Virtual Power Plant* (VPP). A VPP consists of an aggregation of several distributed energy resources, possibly in coalition with conventional generators, with the goal of participating in the energy market as a reliable and more profitable supply source [7]. In our model, the VPP consists of storage and both renewable and conventional generators (Figure 1). The VPP commits to supplying a certain amount of power in the market over a certain time period, and its deviations from its commitment are penalized according to a predetermined pricing scheme imposed by the system operator or the regulator. The VPP's problem is then how to optimally utilize the storage in order to minimize the expected cost of these deviations in the long run. This coincides with our notion of reliability maximization.

The second interpretation of the model corresponds to an abstract model of an energy system as a whole. In this case, the storage is owned and operated by the system operator whose objective is to maximize the reliability of the entire system. The storage in this scenario may be interpreted as a physical component (e.g., battery, flywheel, or pumped hydro) connected to the grid, or, the system inertia, i.e., the kinetic energy of the conventional generators in an energy system. In section 2, we will present a problem formulation that abstractly encompasses all these interpretations.

Recent works [9], [13] have examined the economic value of ramp-constrained storage as an arbitrage mechanism in the energy market. Several other works have used similar abstract models of storage in different settings, to examine optimal/sub-optimal policies and the associated value of ramp-constrained storage in the presence of renewable resources from a combination of economic, reliability, and welfare efficiency perspectives [23], [24], [4] and [14]. Herein, our focus is on reliability (and risk), acknowledging that the economic and reliability values are not completely separable. By abstracting away factors such as the environment, the amount of renewable energy curtailed, price of energy or cost of storage, we characterize the value of storage purely from reliability and risk perspectives. A limitation of our model, however, is the absence of power flow and transmission line constraints, which in practice, contribute significantly to system reliability and the feasibility of satisfying the demand.

Prior research on sizing and characterization of the value of storage for system reliability in energy systems has been reported in [17], [18], [24], [22], [5], [6] and [12]. Particularly, [12] takes a time-series analysis approach to demonstrate that a combination of wind, natural gas and fast ramping energy storage can be used to generate regulated levels of power at a reasonable cost. Also related to our model is the production-storage system studied in [15], where the authors characterize the relaiability of the system under myopic control policies with applications in petrochemical industry. Our paper and [5] and [6] use similar queueing models, though our work is different in many ways. We assume that the storage capacity is fixed and find the optimal policy for withdrawing from storage, as opposed to fixing the policy and optimizing capacity. Another difference is that our model of uncertainty is a compound poisson process instead of the brownian motion model used in [5], [6]. Similar models and concepts exist in the queueing theory literature [11], [16], though with different application contexts.

We formulate the problem of optimal storage management as the problem of minimization of the CED metric, and provide several characterizations of the value function, as well as the structural properties of the associated optimal policy. We prove that for a linear stage cost, a *myopic* policy that drains storage to compensate for all shocks, regardless of their size, is optimal. However, for nonlinear stage costs where the penalty for larger energy deficits is significantly higher, the optimal policy allows for more frequent small energy deficits in order to avoid large energy deficits. Our results show that due to the ramp constraint, the value of storage saturates quickly as a function of capacity, and even more quickly for higher levels of volatility. Furthermore, for a given fixed capacity of storage, the value decreases as volatility increases. Finally, we investigate the effect of storage size on the probability of large energy deficits. We observe that for all control policies, there appears to be a critical level of storage size, above which the probability of large energy deficits diminishes quickly.

Notation 1 *Throughout the paper,* $\mathbb{I}_A$ *denotes the indicator function of a set* A*. The operator* $[x]^+ = \max\{0, x\}$ *is the projection operator onto the nonnegative orthant.*

2 The Model

The details of the model are outlined below.

2.1 Generation

2.1.1 Conventional Generator

The output of the conventional generator is assumed to be controllable, and the controlled generation process is denoted by $\mathbf{G} = \{G_t : t \geq 0\}$, where G_t is the power output at time $t \geq 0$. It is assumed that the power output is subject to an upward ramp constraint and cannot increase instantaneously:

$$\frac{G_t - G_{t'}}{t - t'} \leq \zeta, \qquad \forall t : 0 \leq t < t'.$$

In this paper, we are interested only in energy deficits for serving the load, not in energy excesses from too much production. Therefore, we do not assume a downward ramp constraint on G_t.

2.1.2 Renewable Generator

The renewable generation process is denoted by $\mathbf{R} = \{R_t : t \geq 0\}$. It is assumed that $\mathbf{R}$ can be modeled as a process with two components: $\mathbf{R} = \overline{\mathbf{R}} + \Delta\mathbf{R}$, where $\overline{\mathbf{R}} = \{\overline{R}_t : t \geq 0\}$ is a deterministic process representing the predicted renewable supply, and $\Delta\mathbf{R} = \{\Delta R_t : t \geq 0\}$ is the residual supply assumed to be a random arrival process. Thus, at any given time $t \geq 0$, the total forecast supply from the renewable and controllable generators is given by $G_t + \overline{R}_t$.

2.2 Demand

The demand process is denoted by $\mathbf{D} = \{D_t : t \geq 0\}$, where D_t is the total power demand at time t, assumed to be exogenous and inelastic. Similar to the renewable supply, $\mathbf{D}$ has two components: $\mathbf{D} = \overline{\mathbf{D}} + \Delta\mathbf{D}$, where $\overline{\mathbf{D}} = \{\overline{D}_t : t \geq 0\}$ is the predicted demand process (deterministic), and $\Delta\mathbf{D} = \{\Delta D_t : t \geq 0\}$ is the residual demand, again, assumed to be a random arrival process.

2.3 Storage

The storage process is denoted by $\mathbf{s} = \{s_t \in [0, \overline{s}] : t \geq 0\}$, where s_t is the amount of stored energy at time t, and $\overline{s} < \infty$ is the storage capacity. The storage technology is subject to an upward ramp constraint:

$$\frac{s_t - s_{t'}}{t - t'} \leq r, \qquad \forall t : 0 \leq t < t'.$$

Thus, storage can be used to supply a large amount of energy instantaneously, though the consumed energy cannot be restored instantaneously.

2.4 Problem Formulation

In this section we present the problem formulation in terms of an energy imbalance process[2]. Before we proceed, we introduce a few assumptions and definitions.

Assumption 1 *The forecast supply is equal to the forecast demand. That is:* $\overline{D}_t = G_t + \overline{R}_t$, *for all* $t \geq 0$.

Assumption 1 is a natural assumption which states that the forecast demand is equal to sum of the forecast supply from the renewable resource and the deterministically scheduled supply from the conventional generator. Any imbalance in the system would come from the real-time deviations of demand and renewable generation processes from their forecast, which may be compensated from storage.

Definition 1 The *power imbalance* is defined as the residual demand minus the residual supply:

$$\Delta P_t = \Delta D_t - \Delta R_t. \tag{1}$$

The *normalized energy imbalance* is defined as:

$$W_t = \frac{\Delta P_t^2}{2\zeta}. \tag{2}$$

Assumption 2 *The normalized energy imbalance process* (2) *is the jump process in a compound poisson process with arrival rate* Q *and jump size distribution* f_W, *where the support of* f_W *lies within a bounded interval* $[0, B]$, *where* B *is the maximum deficit that might be incurred in a single shock.*

[2] We refer to the event of not meeting the demand as an energy deficit event, and will use the terms energy deficit and energy imbalance interchangeably.

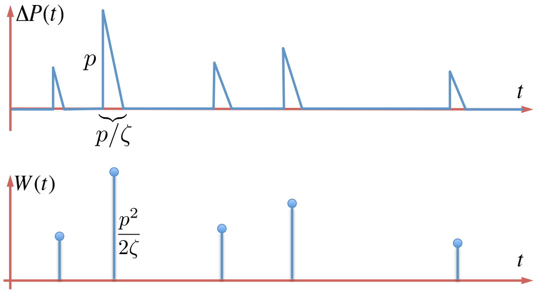

Fig. 2 In the absence of storage, a power imbalance of ΔP_t results in an energy shock of $W_t = \Delta P_t^2/2\zeta$, due to the ramp constraint ζ of the conventional generator. It is assumed that the energy imbalance process can be approximated with a compound Poisson process.

Remark 1 The main intuition behind adopting Assumption 2 for modeling deviations from the nominal (or predicted) trajectory is that smooth deviations can be tracked by the embedded power systems controls (e.g. primary and secondary control) very well. It is the abrupt and large shocks (e.g. loss of a generator or an impulsive change in renewable generation/demand) that is not predictable, cannot be tracked/compensated by slow conventional generation, and requires a fast response that storage could provide. A compound Poisson process is one way to model such phenomena. Compound Poisson processes or their variations have been used in the energy systems literature for modeling electricity price processes in power networks (see, e.g., [21], pages 150 and 216, and the references therein). If we accept prices as a reasonable proxy for the underlying state of the system, these models provide another justification for using compound Poisson processes as a reasonable model for impulsive disturbances that affect the system.

The objective is to design a feasible control policy $\mu : [0, \bar{s}] \times [0, B] \mapsto [0, \bar{s}]$, which maps the state of charge and shock size to the amount of energy to be withdrawn from storage, to maximize the system's reliability. Under Assumptions 1 and 2, the dynamics of the storage process can be written as:

$$s_t = s_0 + \int_0^t \mathbb{I}_{\{s_\tau < \bar{s}\}} r d\tau - \int_0^t \mu\left(s_{\tau^-}, W_\tau\right) dN_\tau \tag{3}$$

where N_t is a Poisson process of rate Q, and W_t is the jump size (energy imbalance) process, drawn independently and identically from a distribution f_W. The term s_{τ^-} denotes the left limit of the storage process at time τ. Here, both N_t and W_t are assumed to be cádlág, i.e., right continuous with left limits. Since the Poisson process model of the energy imbalance process is stationary and memoryless, we focus on stationary Markov policies. We denote the set of all feasible stationary Markovian policies by Π, noting that any feasible policy $\mu(s, w)$ must satisfy $\mu(s, w) \leq s$.

Let $C_\mu(s)$ denote the expected discounted cost of energy deficits under control policy μ, and starting from an initial state s:

$$\begin{aligned} C_\mu(s) &= \mathbf{E}\Big[\int_0^\infty e^{-\theta\tau} g\left(W_\tau - U_\tau\right) dN_\tau \Big| s_0 = s\Big] \\ &= \mathbf{E}\Big[\sum_{k=1}^\infty e^{-\theta t_k} g\left(W_k - \mu(s_{t_k^-}, W_k)\right) \Big| s_0 = s\Big], \end{aligned} \tag{4}$$

where t_k is the k-th Poission arrival time, and $W_k = W_{t_k}$ is size of the k-th jump, and $\theta > 0$ is the discount rate. Moreover, $g : [0, B] \to \mathbb{R}$ is the stage cost as a function of energy imbalance. In this paper, we assume that the stage cost function satisfies certain properties as described in Assumption 3.

Assumption 3 *The stage cost function $g(\cdot)$ is bounded, strictly increasing and continuously differentiable. Moreover, $\mathbf{E}[g(W)] > 0$, and $g(0) = 0$.*

The reliability maximization problem can now be formulated as an infinite horizon stochastic optimal control problem

$$C_\mu(s) \to \min_{\mu \in \Pi} \tag{5}$$

where the optimization problem (5) is subject to the state dynamics (3). A policy $\mu^* \in \Pi$ is defined to be optimal if

$$\mu^* \in \arg\min_{\mu \in \Pi} C_\mu(s).$$

The associated *value function* or optimal cost function is denoted by $C(s)$:

$$C(s) = \min_{\mu \in \Pi} C_\mu(s), \quad 0 \le s \le \bar{s}. \tag{6}$$

3 Characterization of The Value Function and the Associated Optimal Policy

3.1 Characterizations of the Value Function

We first provide several characterizations for the value function defined in (6) and establish specific properties that are useful in characterization of the optimal policy.

Let $J_\mu(s, w)$ be the expected cost under policy μ, conditioned on the first jump arriving at time $t_1 = 0$, and being of size w. Here, s is the state of the system just before executing the action dictated by the policy. By the memoryless property of the Poisson process, we have

$$\begin{aligned} &J_\mu(s, w) = g(w - \mu(s, w)) \\ &+\mathbf{E}\Big[\sum_{k=1}^\infty e^{-\theta t_k} g(W_k - \mu(s_{t_k^-}, W_k))\Big| s_0 = s - \mu(s, w)\Big] \end{aligned} \tag{7}$$

We may relate $J_\mu(s, W)$ to the total expected cost $C_\mu(s)$ defined in (4) as follows:

$$C_\mu(s) = \mathbf{E}\left[e^{-\theta t_0} J_\mu(\min\{s + rt_0, \bar{s}\}, W)\right], \tag{8}$$

where t_0 is an exponential random variable with mean $1/Q$, and is independent of W, drawn from distribution f_W.

From (8), it is clear that by minimizing J_μ (given in (7)) across all admissible policies, we can obtain the optimal solution to the original problem (5). The discrete-time formulation of J_μ given in (7) facilitates deriving the necessary and sufficient optimality condition, as well as development of efficient numerical methods. We summarize these results in the following theorem.

Theorem 1 *Given an admissible control policy $\mu \in \Pi$, let $J_\mu : [0, \bar{s}] \times [0, B] \mapsto \mathbb{R}$ be the function defined in (7). A function $J : [0, \bar{s}] \times [0, B] \mapsto \mathbb{R}$ satisfies*

$$J(s, w) = J^*(s, w) \stackrel{\text{def}}{=} \min_{\mu \in \Pi} J_\mu(s, w), \qquad \forall (s, w),$$

if and only if it satisfies the following fixed-point equation:

$$\begin{aligned} J(s, w) = (TJ)(s, w) \stackrel{\text{def}}{=} \min_{u \in [0, \min\{s, w\}]} \Big\{ & g(w - u) \\ & + \mathbf{E}\Big[e^{-\theta t_0} J\big(\min\{s - u + rt_0, \bar{s}\}, W\big)\Big]\Big\}. \end{aligned} \tag{9}$$

Moreover, a stationary policy $\mu^*(s, w)$ is optimal if and only if $u = \mu^*(s, w)$ achieves the minimum in (9) for $J = J^*$. Finally, the value iteration algorithm

$$J_{k+1} = TJ_k, \tag{10}$$

converges to J^* for any initial condition J_0.

Proof The result follows from establishing the contraction property of T, which is straightforward in this case due to bounded stage and discounted cost. See [3] for more details.

While Theorem 1 provides the basis for numerical computation of the optimal cost function and optimal policy, we can derive further analytical characterizations of the optimal policy based on continuous-time analysis of problem (5), which leads to the Hamilton-Jacobi-Bellman (HJB) equation [20]. In the following theorem, we present some basic properties of the optimal cost function. We present the proofs of the following theorems in the Appendix.

Theorem 2 *Let $C(s)$ be the optimal cost function defined in (6). The following statements hold:*

(i) $C(s)$ is strictly decreasing in s.

(ii) If the stage cost $g(\cdot)$ is convex, the optimal cost function $C(s)$ is also convex in s.

(iii) If $C(\cdot)$ is continuously differentiable, then for all $s \in [0, \bar{s}]$, it satisfies the following HJB equation

$$\frac{dC(s)}{ds} = \frac{Q+\theta}{r} C(s) - \frac{Q}{r} \mathbf{E}\Big[\min_{u \in [0, \min\{s, W\}]} g(W-u) + C(s-u) \Big], \quad (11)$$

with the boundary condition

$$\frac{dC}{ds}\Big|_{s=\bar{s}} = 0. \quad (12)$$

Moreover, the optimal policy $\mu^*(s,w)$ achieves the optimal solution of the minimization problem in (11). Furthermore, for a given policy μ, if the cost function $C_\mu(s)$ is differentiable, it satisfies the following differential equation

$$\begin{aligned} \frac{dC_\mu}{ds} &= \frac{Q+\theta}{r} C_\mu(s) \\ &\quad - \frac{Q}{r} \mathbf{E}\Big[g(W - \mu(s,W)) + C_\mu(s - \mu(s,W)) \Big], \end{aligned} \quad (13)$$

with the boundary condition (12).

The result of Theorem 2 part (iii) requires continuous differentiability of the optimal cost function, which can be established under some mild conditions such as differentiability of the stage cost function g and the probability density function $f_W(\cdot)$ of Poisson jumps (cf. Benveniste and Scheinkman [2]). Throughout this paper, we assume that $C(s)$ is continuously differentiable and the results of Theorem 2 are applicable.

3.2 Characterizations of the Optimal Policy

In this subsection, we derive some structural properties of the optimal policy using the optimal cost characterizations given in Theorems 1 and 2. First, we show that the myopic policy of allocating reserve energy from storage to cover as much of every shock as possible is optimal for linear stage cost functions. Then, we give a partial characterization of the structure of the optimal policy for strictly convex stage cost functions.

Theorem 3 *If the stage cost is linear, i.e., $g(x) = \beta x$ for some $\beta > 0$, then the myopic policy*

$$\mu^*(s,w) = \min\{s, w\}, \quad (14)$$

is optimal for problem (5).

We will see in Section 4 that the myopic policy (14) results in a higher probability of large energy deficits. Intuitively, the myopic policy greedily consumes the reserve, and thereby increases the chances of having little or no energy in reserve when facing a large supply shock. In the linear stage cost case, the penalty for a large energy deficit is equivalent to the total penalty of many small energy deficits. This is unlike the strictly convex case. Therefore, the optimal policy for the strictly convex case tends to be more conservative in consuming the reserve.

Next, we focus on strictly convex stage cost functions, and present some characterizations of the structural properties of the optimal policy using the results derived within Section 3.1. Before we proceed, we introduce a definition.

Definition 2 The *drift* of the storage process is the ramp rate of storage minus the rate of the compound Poisson process, and is denoted by δ:

$$\delta = r - Q\mathbf{E}[W]. \tag{15}$$

Theorem 4 *Let $\mu^*(s,w)$ be the optimal policy associated with problem* (5)*. If $\delta \geq 0$, i.e., if the storage process has nonnegative drift, then $\mu^*(s,w)$ is monotonically nondecreasing in both s and w.*

Theorem 5 *Let $\mu^*(s,w)$ denote the optimal policy associated with problem* (5) *with strictly convex stage cost $g(\cdot)$. There exists a unique kernel function $\phi : [-B, \bar{s}] \to \mathbb{R}$ such that*

$$\mu^*(s,w) = \Big[w - \phi(s-w)\Big]^+, \quad \forall (s,w) \in [0,\bar{s}] \times [0,B], \tag{16}$$

where,

$$\begin{aligned} \phi\,(p) = \arg\min_x \; & g\,(x) + C\,(x+p) \\ \text{s.t. } & x \leq \min\{B, \bar{s} - p\} \\ & x \geq \max\{0, -p\} \end{aligned} \tag{17}$$

Moreover, if $\delta \geq 0$, the kernel function $\phi(\cdot)$ can be represented as follows:

$$\phi(p) = \begin{cases} -p, & -B \leq p \leq b_0 \\ \phi^\circ(p), & b_0 \leq p \leq b_1 \\ 0, & b_1 \leq p \leq \bar{s}, \end{cases} \tag{18}$$

where $\phi^\circ(p)$ is the unique solution of

$$g'(x) + C'(x+p) = 0, \tag{19}$$

and b_0 and b_1 are the break-points, where

$$b_0 \;=\; -(g')^{-1}\Big(\;-\;C'(0)\Big) \;\geq\; -(g')^{-1}\Big(\frac{Q}{r}\mathbf{E}[g(W)]\Big) \;\geq\; -B, \tag{20}$$

$$b_1 = (C')^{-1}\Big(-g'(0)\Big) \leq \bar{s}. \tag{21}$$

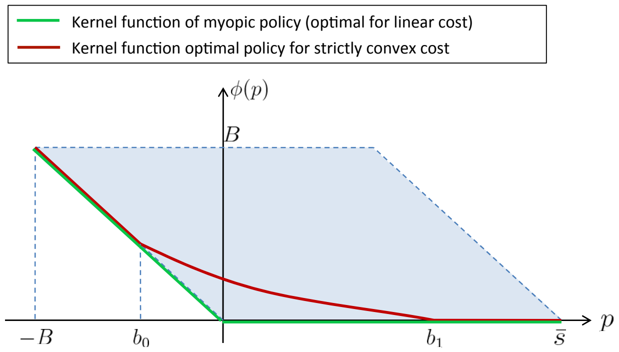

Fig. 3 Structure of the kernel function $\phi(p)$ defined in (17).

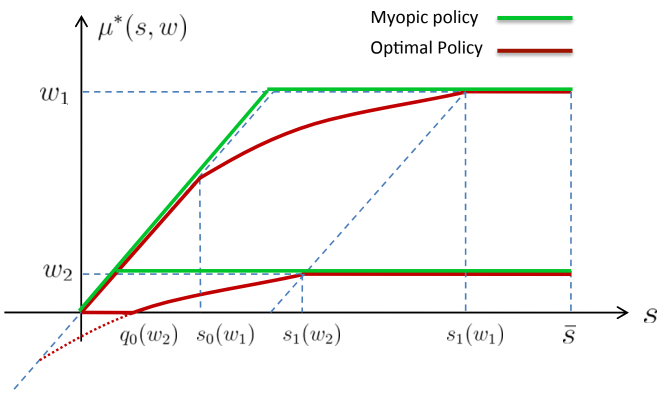

Fig. 4 Structure of the optimal policy $\mu^*(s,w)$ for a convex stage cost, for two different shock sizes w_1 and w_2, where $w_2 < w_1$. For comparison, the myopic policy is shown in green.

Theorem 5 reveals a very special structure for the optimal policy. In fact, it shows that the two dimensional policy can be represented using a single dimensional kernel function. This result allows us to significantly reduce the computational complexity of various simulations presented in Section 4. In addition, using Theorem 5, we can provide a qualitative picture of the structure of the optimal policy. Figures 3 and 4 illustrate a conceptual plot of the kernel function, and the optimal policy, respectively.

In particular, for all $(s, w) \in [0, \bar{s}] \times [0, B]$, we can summarize the characterization of the optimal policy as follows. If $w \geq -b_0$, we have

$$\mu^*(s,w) = \begin{cases} s, & 0 \leq s \leq s_0(w) \\ w - \phi^\circ(s-w), & s_0(w) \leq s \leq s_1(w) \\ w, & s_1(w) \leq s, \end{cases} \tag{22}$$

where $s_i(w) = w + b_i$ for $i = 0, 1$. For $w \leq -b_0$, we have

$$\mu^*(s,w) = \begin{cases} 0, & 0 \leq s \leq q_0(w) \\ w - \phi^\circ(s-w), & q_0(w) \leq s \leq s_1(w) \\ w, & s_1(w) \leq s, \end{cases} \tag{23}$$

where $q_0(w)$ is the unique solution of $\phi^\circ(s-w) = w$. Note that $\mu^*(s,w)$ may never achieve the value w, if $s_1(w) > \bar{s}$.

The interpretation of this optimal policy which is schematically shown in Figure 4 is as follows. First, it is possible to have $\mu^*(s,w) < \min\{s,w\}$, implying that it is better to afford a small deviation penalty in the present to avoid a large energy deficit in the future. On the other hand, we observe that $\mu^*(s,w) = w$ for $s > s_1(w)$. This makes intuitive sense since it implies that if there is sufficiently large reserve in the system, it is optimal to cover the supply shock by withdrawing from storage. For a relatively small shock size (e.g., w_2 in Figure 4), no action will be taken, unless the storage state is above a certain threshold $q_0(w_2)$. For a larger shock size (e.g., w_1), we have $q_0(w_1) = 0$, implying that always some energy will be withdrawn from storage, regardless of its state. It is optimal to fully drain the storage up to another threshold $s_0(w1)$. The most interesting case is exactly the middle range (between $q_0(w)$ and $s_1(w)$ for small shocks and between $s_0(w)$ and $s_1(w)$ for larger shocks). In this regime, we have $0 < \mu^*(s,w) < \min\{s,w\}$, meaning that we accept partial energy deficit so as to not drain the storage too much, and be left more vulnerable to future shocks.

4 Numerical Simulations

In this section, we use numerical simulations to study the effect of storage size and volatility on system reliability and risk. We use the value iteration algorithm (10) to compute the optimal policy and cost function for nonlinear stage costs. Figures 5 illustrates the optimal policy in a scenario with uniformly distributed random jumps and quadratic stage cost. Note that the structure of the actual optimal policy (Figure 5) conforms with the structure derived based on analytical results of Theorem 5, shown in Figure 4. The computational procedure for finding the optimal policy is used for further simulations discussed below.

4.1 Reliability Value

Figures 6 and 7 show the value of storage—defined as the normalized improvement in expected cost as a function of storage size—for different Poisson arrival rates. In these simulations we have $\theta = 0.01$, $r = 1$, and $W = 1$. In Figure 7 we have a cubic stage cost, while in Figure 6 the stage cost is quadratic. We observe that while adding a small amount of storage improves reliability significantly, the value saturates quickly as a function of storage

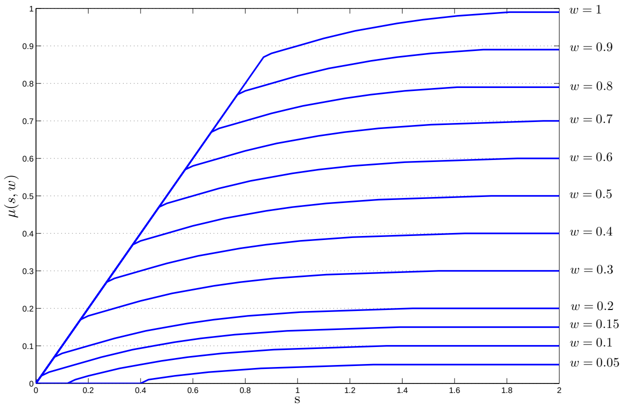

Fig. 5 Opt iteration algorithm (10) for quadratic stage cost, uniform shock distribution, and the following parameters: $\theta = 0.1, r = 1, Q = 0.8, \bar{s} = 2$. The horizontal axis is the storage state and the vertical axis is the amount of energy to be withdrawn in response to a shock. The different curves correspond to different shock sizes.

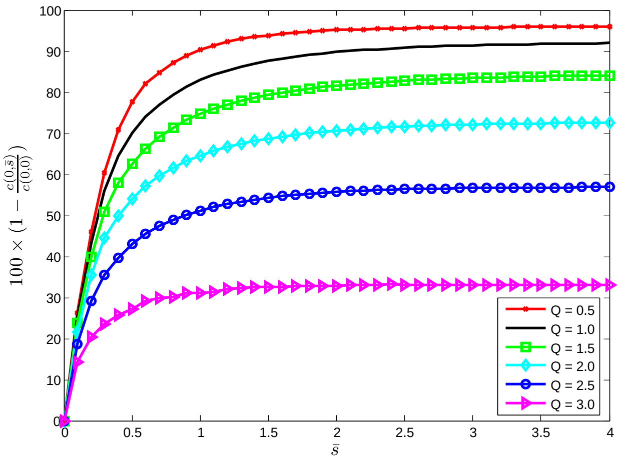

Fig. 6 Value of storage as a function of the storage capacity for cubic stage cost, different Poisson arrival rates, $r = 1$, and deterministic jumps size $W = 1$. $c(s; \bar{s})$ denotes the optimal cost function $c(s)$ (c.f. Equation (6)) when the storage capacity is given by $\bar{s}$.

capacity. Moreover, the value of storage is lower for higher levels of volatility, i.e., high arrival rates of shocks. Similar behavior is observed in extensive simulations that point us to the qualitative conclusions that a relatively small amount of storage provides most of the reliability value, and that as volatility in the system increases, the reliability value of storage decreases. Beyond a certain limit of storage capacity, it is only faster storage technology that helps improve reliability, not additional storage capacity of the same technology.

To examine the effects of different ramp rates, and different levels of volatility on reliability value of storage, we performed another set of simulations with

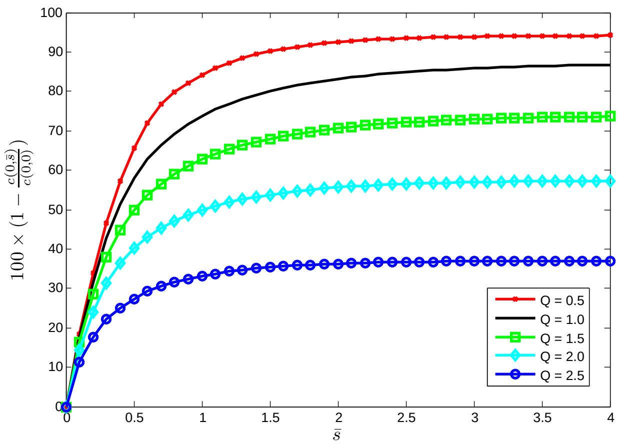

Fig. 7 Value of storage as a function of the storage capacity for quadratic stage cost, different Poisson arrival rates, $r = 1$, and deterministic jumps size $W = 1$. $c(s;\bar{s})$ denotes the optimal cost function $c(s)$(c.f. Equation (6)) when the storage capacity is given by $\bar{s}$.

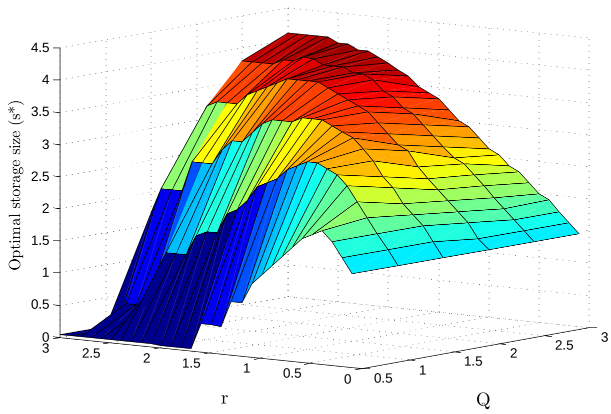

Fig. 8 Simulation results for optimal sizing. Here, we have deterministic shocks with jump size $W = 1$, while s^* denotes the capacity that achieves 90% of the reliability value. We observe that s^* is largest when $Q = r$, i.e., when the storage has zero drift: $\delta = r - Q\mathbf{E}[W] = 0$.

the results presented in Figures 8 and 9. We define s^* to be the amount of storage capacity that achieves 90% of the maximum reliability value, and examine the value of s^* as a function of system parameters. In the first case (Figure 8), we set $W = 1$ and examine s^* as a function of the arrival rate Q and ramp rate r. In the second case (Figure 9), we set $r = 1$ and examine s^* as a function of the arrival rate Q and shock size W. The interesting observation is that maximum storage capacity corresponds to the case where the storage has zero drift: $\delta = r - Q\mathbf{E}[W] = 0$. In other words, when we are optimizing storage capacity in order to improve system reliability, systems with zero drift

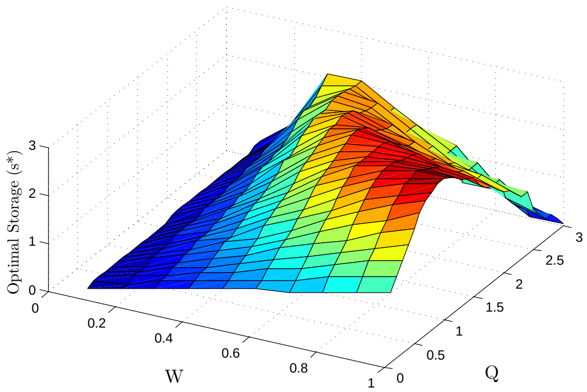

Fig. 9 Simulation results for optimal sizing. Here, we have the ramp rate $r = 1$, while s^* denotes the capacity that achieves 90% of the reliability value. We observe that s^* is largest when $QW = 1$, i.e., when the storage has zero drift: $\delta = r - Q\mathbf{E}[W] = 0$.

would be associated with the highest amount of storage capacity. The intuition is as follows. If the drift is positive, volatility is relatively low and a smaller amount of storage achieves most of the value, because, in some sense, storage is always full and available. On the other hand, when the drift is negative, the value of more capacity diminishes because of high volatility which prevents restoring energy. This has a strong implication for system design. Optimal sizing of storage should be for a target level of volatility. If volatility changes drastically from the target volatility then the value of storage capacity may decrease significantly.

4.2 Energy Deficit Statistics and Risk

In this section, we study the effect of different optimal policies associated with different stage costs on the probability distribution of energy deficits, and particularly, the risk of large energy deficits. Figure 10 shows the energy imbalance distribution in a scenario with deterministic jumps of unit size, for both myopic policy and the optimal policy for a quadratic cost function. Figure 11 shows the energy deficit distribution for the same system but for cubic stage cost. Note that, the stage cost for the non-myopic policy assigns a significantly higher weight to larger energy deficits. Therefore, as we can see in Figures 10 and 11, the non-myopic policy results in less frequent large energy deficits at the cost of more frequent small energy deficits. The same trend is observed between the statistics of energy deficit events under quadratic and cubic stage costs. In conclusion, the rate of growth of the stage cost determines the statics of the energy deficits. The faster the growth of the cost as a function of storage size, the smaller the probability of large energy imbalances will be. This reduction in risk is achieved at the cost of a higher probability of small energy deficits.

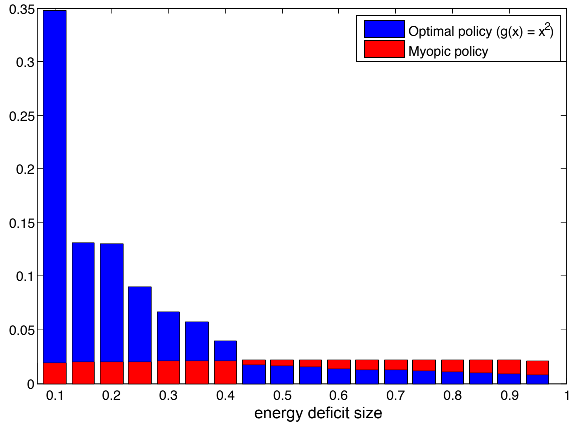

Fig. 10 Energy deficit distribution comparison of myopic and non-myopic policies for quadratic stage costs. Deterministic jumps with $W = 1$ and rate $Q = 0.8$.

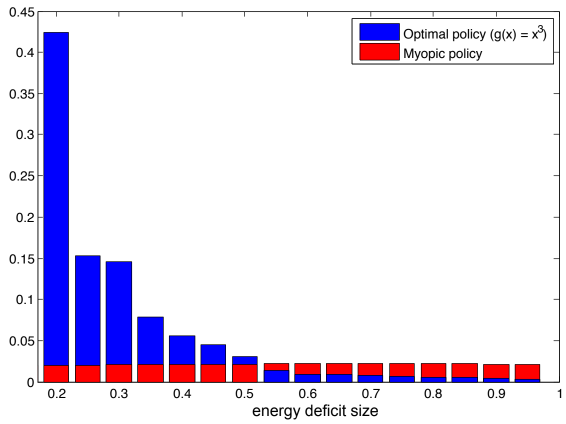

Fig. 11 Energy deficit distribution comparison of myopic and non-myopic policies for cubic stage cost. Deterministic jumps with $W = 1$ and rate $Q = 0.8$.

Finally, we look at the effect of storage size on the risk of incurring large energy deficits. Figure 12 plots this risk metric for different policies that are all optimal for different stage cost functions. Here, we observe a sharp improvement in risk mitigation at a critical level of storage capacity. This critical level is higher (near the maximum shock size) for the myopic policy and much lower (near the average shock size) for the quadratic and cubic policies. This is consistent with the previous simulations on the reliability value, and leads to the qualitative conclusion that a relatively small amount of storage provides most of the value both in terms of improving reliability and mitigating risk.

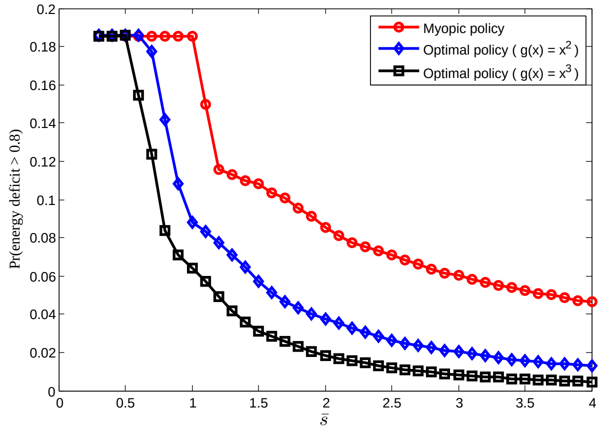

Fig. 12 Probability of large energy deficits as a function of storage size for different policies (uniformly distributed jump $W = U[0, 1]$ and rate $Q = 1$).

5 Summary and Concluding Remarks

We examined the value of storage in a system with uncertainty in supply/demand, and upward ramp constraints for both conventional generation and charging of storage. The uncertainty was modeled as a compound poisson arrival of energy deficit shocks. We formulated the problem of maximization of system reliability as the problem of minimization of the infinite horizon expected discounted cost of energy deficits over all stationary Markovian policies. We showed that for a linear stage cost, a *myopic* policy which uses storage to compensate for all shocks regardless of their size is optimal. However, for strictly convex stage costs it maybe optimal to incur a small energy imbalance in order to avoid a large energy deficits in the future. An interpretation of this result is that attempting to mitigate all small energy deficits increases the probability of large energy deficits. We also showed that the reliability value of storage saturates quickly as a function of storage capacity, and beyond a certain level, it is only faster storage that can bring more value for the system, not more capacity with the same technology. Another interesting observation is the sensitivity of the optimal storage size that achieves *most* of the reliability value to the drift of the storage process. In a system with zero drift, storage is of the most value and more capacity will be integrated in an optimally designed system. However, deviations from the zero drift scenario drastically diminish the value of storage capacity. Thus, optimal sizing of storage must be based on a carefully calculated target level of volatility. Finally, Our results suggest that for all control policies, there seems to be a critical level of storage size, above which the probability of suffering large energy deficits diminishes quickly. This level is higher for the myopic policy and lower for optimal policies with strictly convex stage costs.

Our results have important implications on different aspects of planing, design, and operation of energy systems. For instance, in designing market mechanisms for Virtual Power Plants (VPPs), the system operator can expect infrequent but large deviations from the scheduled output of VPP if the output deviation penalty is linear in the size of deviation, whereas, a pricing mechanism that grows nonlinearly in the size of the deviation, will result in more frequent but smaller deviations. The system operator can then choose the desired tradeoff based on the available reserve technologies, and their cost. An interesting direction for future research would be to model and examine the effects of power flow and transmission network constraints on the value of storage.

A Appendix

Proof of Theorem 2: *Part (i):* The monotonicity property of the value function follows almost immediately from the definition. Let $0 \leq s_1 < s_2 \leq \bar{s}$, and assume $C(s) = C_\mu(s)$ for some policy μ. Given the initial state s_1, let $u_t^{(1)}$ be the control process under policy μ. Note that for every realization ω of the compound Poisson process, the sample path $u_t^{(1)}(\omega)$ is admissible for initial condition $s_2 > s_1$. Therefore, by definitions (4) and (6), we have $C(s_2) \leq C(s_1)$.

In order to show the strict monotonicity, consider the controlled process starting from s_1. Let τ be the first arrival time such that $g(W_\tau - u_\tau^{(1)}) > 0$. By Assumption 3, we have $\mathbf{P}(\tau \in [0,T]) > 0$ for some $T < \infty$. For every sample path ω, define the control process

$$u_t^{(2)}(\omega) = u_t^{(1)}(\omega) + \delta \cdot \mathbb{I}_{\{t=\tau(\omega)\}},$$

for some $\delta > 0$ such that $\delta \leq \min\{s_2 - s_1, W_{\tau(\omega)} - u_{\tau(\omega)}^{(1)}\}$.

It is clear that $u_t^{(2)}(\omega)$ is admissible for the controlled process starting from s_2. Using the definition of the expected cost function in (4), we can write

$$\begin{aligned} C(s_1) - C(s_2) &\geq \mathbf{E}_\omega[e^{-\theta\tau(\omega)} g(W_{\tau(\omega)} - u_{\tau(\omega)}^{(1)}) - e^{-\theta\tau(\omega)} g(W_{\tau(\omega)} - u_{\tau(\omega)}^{(1)} - \delta)] \\ &\geq \mathbf{E}[\epsilon e^{-\theta\tau(\omega)}], \quad \text{for some } \epsilon > 0 \\ &\geq \epsilon e^{-\theta T} \mathbf{P}(\tau \in [0,T]) > 0, \end{aligned}$$

where the first inequality holds by strict monotonicity of g.

Part (ii): We first prove convexity of $J^*(s,w)$ defined in Theorem 1, and use it to establish convexity of $C(s)$.

In order to show convexity of $J^*(s,w)$, we need to show that the operator T defined in (9) preserves convexity. Then the claim would be immediate using the convergence of value iteration algorithm (10) to optimal cost J^*, where the initial condition is an arbitrary convex function such as $J_0 = 0$.

Next we show that the operator T preserves convexity for this particular problem. Define the objective function in (9) as $Q(s,w,u)$. We have

$$\begin{aligned} Q(s,w,u) &= g(w-u) + \mathbf{E}\Big[e^{-\theta t_0} J\big(\min\{s-u+rt_0, \bar{s}\}, W\big)\Big] \\ &= g(w-u) + \int_{\frac{\bar{s}-s+u}{r}}^{\infty} e^{-\theta t_0} \mathbf{E}[J\big(\bar{s}, W\big)] R e^{-Qt_0} dt_0 \\ &\quad + \int_0^{\frac{\bar{s}-s+u}{r}} e^{-\theta t_0} \mathbf{E}[J\big(s-u+rt_0, W\big)] R e^{-Qt_0} dt_0. \end{aligned}$$

Using the fact that J is convex, linearity of expectation and basic definition of a convex function, it is straightforward but tedious to show that $Q(s, w, u)$ is a convex function. We omit the details for brevity. Given the convexity of Q, the convexity of $(TJ)(s, w)$ is immediate, since we are minimizing a multidimensional convex function over one of its dimensions. Hence, we have established convexity of $J^*(s, w)$ in (s, w). Finally, we can express $C(s)$ in terms of $J^*(s, w)$ as in (8). This results in convexity of $C(s)$ using the above argument for proving convexity of $Q(s, w, u)$.

Part (iii): The derivation of Hamilton-Jacobi-Bellman is relatively standard. We omit the proof for brevity. For a more detailed treatment, please refer to [3], [10] and [8]. ∎

Proof of Theorem 3: We establish optimality of μ^* by showing that it achieves an expected cost no higher than any other admissible policy. Consider an admissible policy $\tilde{\mu}$ such that $\tilde{\mu}(s, w) < \min\{s, w\}$ for some $(s, w) \in [0, \bar{s}] \times [0, B]$. For every sample path of the controlled process, let $\tau_1(\omega)$ be the first Poisson arrival time such that

$$\min\{s_{\tau_1^-}, W_{\tau_1}\} - \tilde{\mu}(s_{\tau_1^-}, W_{\tau_1}) = \epsilon > 0.$$

Therefore, by applying policy $\tilde{\mu}$ instead of μ^*, we pay an extra penalty of $\beta\epsilon e^{-\theta\tau_1(\omega)}$. The reward for this extra penalty is that the state process is now biased by at most ϵ, which allows us to avoid later penalties. However, since the stage cost is linear, the penalty reduction by this bias for any time $\tau_2(\omega) > \tau_1(\omega)$ is at most $\beta\epsilon e^{-\theta\tau_2(\omega)}$. Hence, for this sample path ω, the policy $\tilde{\mu}$ does worse than the myopic policy μ^* at least by $\beta\epsilon(e^{-\theta\tau_1(\omega)} - e^{-\theta\tau_2(\omega)}) > 0$. Therefore, by taking the expectation for all sample paths, the myopic policy cannot do worse than any other admissible policy. Note that this argument does not prove the uniqueness of μ^* as the optimal policy. In fact, we may construct optimal policies that are different from μ^* on a set $A \subseteq [0, \bar{s}] \times [0, B]$, where $\mathbf{P}((s_{t-}, W_t) \in A) = 0$. ∎

We delay the proof of Theorem 4 until after proof of Theorem 5. Let us start with some useful lemmas on the structure of the kernel function.

Lemma 1 *Let $\phi(p)$ be defined as in (17). We have*

1. *If $\phi(p_0) = -p_0$ for some p_0, then $\phi(p) = -p$, for all $p \leq p_0$.*
2. *If $\phi(p_1) = 0$ for some p_1, then $\phi(p) = 0$, for all $p \geq p_1$.*

Proof By convexity of the stage cost function and Theorem 2(ii), $\phi(p)$ is the optimal solution of a convex program. Therefore, if $\phi(p_0) = -p_0$ for some $p_0 \leq 0$, we have

$$g'(-p_0) + C'(0) \geq 0.$$

Thus, by convexity of stage cost, $g(-p) \geq g(-p_0)$, for any $p \leq p_0$. Therefore, by convexity of $C(\cdot)$ and $g(\cdot)$,

$$g'(x) + C'(x + p) \geq g'(-p) + C'(0) \geq 0, \text{for all } x \geq -p,$$

which immediately implies optimality of $(-p)$, for $p \leq p_0$.

Similarly, for the case where $\phi(p_1) = 0$, we have $g'(0) + C'(p_1) \geq 0$, which implies

$$g'(x) + C'(x + p) \geq g'(0) + C'(p) \geq 0, \quad \text{for all } p \geq p_1,$$

hence, the objective is nondecreasing for all feasible x and $\phi(p) = 0$. ∎

Lemma 2 *Let $C(s)$ be defined as in (6), and assume that the stage cost $g(\cdot)$ is convex. Then*

$$\frac{dC}{ds}(s) \geq -\frac{Q}{r}\mathbf{E}_W[g(W)], \quad 0 \leq s \leq \bar{s}. \tag{24}$$

Proof By Theorem 2(ii), the optimal cost function $C(s)$ is convex. Hence, $\frac{dC}{ds}(s) \geq \frac{dC}{ds}(0)$. On the other hand, by Theorem 2(iii), we can write

$$\frac{dC}{ds}(0) = \frac{Q+\theta}{r}C(0) - \frac{Q}{r}\mathbf{E}_W\Big[\min_{u=0} g(W-0) + C(0)\Big].$$

Combining the two preceding relations proves the claim. ∎

Lemma 3 *If $\delta \geq 0$, then the first constraint in (17) is never active, i.e., $\phi(p) < \min\{B, \bar{s}-p\}$.*

Proof We show that under the non-negative drift assumption, the slope of the objective function is always non-negative at $x = \min\{B, \bar{s}-p\}$. In the case where $\bar{s}-p \leq B$, we have

$$\frac{\partial}{\partial x}\Big(g(x) + C(x+p)\Big)\Big|_{x=\bar{s}-p} = g'(\bar{s}-p) + C'(\bar{s}) \geq 0,$$

where the inequality follows from monotonicity of g and (12). For the case where $\bar{s}-p \geq B$, we employ Lemma 2 and non-negative drift assumption to write

$$\begin{aligned}\frac{\partial}{\partial x}\Big(g(x) + C(x+p)\Big)\Big|_{x=B} &= g'(B) + C'(B+p)\\ &\geq g'(B) - \frac{Q}{r}\mathbf{E}_W[g(W)] \geq g'(B) - \frac{\mathbf{E}_W[g(W)]}{\mathbf{E}[W]} \geq 0,\end{aligned}$$

where the last inequality holds because $g(w) \leq wg'(B)$, for all $w \leq B$, which is a convexity result. ∎

Proof of Theorem 5: By Theorem 2(iii), we can characterize the optimal policy as

$$\begin{aligned}\mu^*(s,w) = \text{argmin} \;\; & g(w-u) + C(s-u)\\ \text{s.t. } & 0 \leq u \leq \min\{s,w\}.\end{aligned} \tag{25}$$

Note that the optimization problem in (25) is convex, because $g(\cdot)$ and hence, $C(\cdot)$ is convex (cf. Theorem 2(ii)). Using the change of variables

$$x = w-u, \quad p = s-w,$$

we can rewrite (25) as $\mu^*(s,w) = w - x^*(p,w)$, where

$$\begin{aligned}x^*(p,w) = \text{argmin} \;\; & g(x) + C(p+x)\\ \text{s.t. } & x \geq \max\{0,-p\}\\ & x \leq w.\end{aligned} \tag{26}$$

The optimization problem in (26) depends on both parameters p and w. We may remove the dependency on w as follows. Since $w \leq B, \bar{s}-p$, we may relax the last constraint, $x \leq w$, by replacing it with $x \leq \min\{B, \bar{s}-p\}$ The optimal solution of the relaxed problem is the same as $\phi(p)$ defined in (17). If $\phi(p) < w$, then the relaxed constraint is not active, and $\phi(p)$ is also the solution of (26). Otherwise, since we have a convex problem, the constraint $x \leq w$ must be active, which uniquely identifies the optimal solution as w. Therefore, the optimal solution of the problem in (26) is given by $x^*(p,w) = \min\{\phi(p), w\}$. Combining the preceding relations, we obtain

$$\mu^*(s,w) = w - \min\{\phi(s-w), w\} = \Big[w - \phi(s-w)\Big]^+.$$

The representation in (18) is a direct consequence of Lemmas 1 and 3. Between some breakpoints b_0 and b_1, the optimal solution of (17) can only be an interior solution, which is given by (19). The uniqueness of $\phi^\circ(p)$ follows from strict convexity of g. Finally, by continuous differentiability of the cost function, equation (19) should hold at the break-points as well. Therefore,

$$g'(-b_0) + C'((-b_0) + b_0) = 0, \quad g'(0) + C'(0 + b_1) = 0,$$

which is equivalent to the characterizations in (20) and (21). The first inequality in (20) holds by Lemma 2 and convexity of $g(\cdot)$, and the second inequality holds by the assumption that $\delta \geq 0$, and applying convexity of $g(\cdot)$ again. ■

Lemma 4 *Let $\phi(p)$ be defined as in (17), and assume that the storage process drift $\delta \geq 0$, and the stage cost $g(\cdot)$ is strictly convex. Then for all $p_1 \leq p_2$,*

$$-(p_2 - p_1) \leq \phi(p_2) - \phi(p_1) \leq 0. \tag{27}$$

Proof We first establish the monotonicity of $\phi(p)$. Let $p_1 < p_2$. Given the structure of the kernel function in (18), there are multiple cases to consider, for most of which the claim is immediate using (18). We only present the case where $-B \leq p_1 \leq b_1$ and $b_0 \leq p_2 \leq b_1$. A necessary optimality condition at p_1 is given by

$$g'(\phi(p_1)) + C'(p_1 + \phi(p_1)) \geq 0. \tag{28}$$

Similarly, for p_2, we must have

$$g'(\phi(p_2)) + C'(p_2 + \phi(p_2)) = 0, \tag{29}$$

Now, assume $\phi(p_2) > \phi(p_1)$. By convexity of $C(\cdot)$ (cf. Theorem 2(ii)) and strict convexity of $g(\cdot)$, we obtain

$$g'(\phi(p_2)) + C'(p_2 + \phi(p_2)) > g'(\phi(p_1)) + C'(p_1 + \phi(p_1)) \geq 0,$$

which is a contradiction to (29).

For the second part of the claim, again, we should consider several cases depending on the interval to which p_1 and p_2 belong. Here, we present the case where $b_0 \leq p_1 \leq b_2$ and $b_0 \leq p_2 \leq \bar{s}$. The remaining cases are straightforward using (18). In this case, we have

$$g'(\phi(p_1)) + C'(p_1 + \phi(p_1)) = 0, \tag{30}$$

$$g'(\phi(p_2)) + C'(p_2 + \phi(p_2)) \geq 0. \tag{31}$$

Combine the optimality conditions in (30) and (31) to get

$$g'(\phi(p_2)) + C'(p_2 + \phi(p_2)) \geq g'(\phi(p_1)) + C'(p_1 + \phi(p_1)) \tag{32}$$

Assume $\phi(p_2) < \phi(p_1)$; otherwise, the claim is trivial. By strict convexity of $g(\cdot)$, we have $g'(\phi(p_2)) < g'(\phi(p_1))$. Therefore by (32), it is true that

$$C'(p_2 + \phi(p_2)) > C'(p_1 + \phi(p_1)). \tag{33}$$

Now assume $\phi(p_2) - \phi(p_1) < -(p_2 - p_1)$. By rearranging the terms of this inequality and invoking the convexity of $C(\cdot)$, we get $C'(p_2 + \phi(p_2)) \leq C'(p_1 + \phi(p_1))$, which is in contradiction to (33). Therefore, the claim holds. ■

Proof of Theorem 4: First, note that by Lemma 4, we get

$$\phi(s_2 - w) \leq \phi(s_1 - w), \quad \text{for all } w, s_1 \leq s_2$$

which implies (cf. Theorem 5)

$$\mu^*(s_2, w) = \left[w - \phi(s_2 - w)\right]^+ \geq \left[w - \phi(s_1 - w)\right]^+ = \mu^*(s_1, w).$$

Moreover, for all s and $w_1 \leq w_2$, we can use the second part of Lemma 4 to conclude

$$\phi(s - w_1) - \phi(s - w_2) \geq -(w_2 - w_1).$$

By rearranging the terms, it follows that

$$\mu^*(s, w_2) = \left[w_2 - \phi(s - w_2)\right]^+ \geq \left[w_1 - \phi(s - w_1)\right]^+ = \mu^*(s, w_1),$$

which completes the proof. ■

References

1. Agency, I.R.E.: Electricity storage: Technology brief. In: ETSAP: Energy Technology Systems Analysis Programme (2012, http://iea-etsap.org/web/Supply.asp)
2. Benveniste, L., Scheinkman, J.: On the differentiability of the value function in dynamic models of economics. Econometrica **47**(3) (1979)
3. Bertsekas, D.: Dynamic Programming and Optimal Control, vol. 1,2, 3rd edn. Athena Scientific (2007)
4. Brown, P., Peas Lopes, J., Matos, M.: Optimization of pumped storage capacity in an isolated power system with large renewable penetration. Power Systems, IEEE Trans. on **23**(2), 523 –531 (2008)
5. Chen, M., Cho, I.K., Meyn, S.: Reliability by design in a distributed power transmission network. Automatica **42**, 1267–1281 (2006)
6. Cho, I.K., Meyn, S.P.: A dynamic newsboy model for optimal reserve management in electricity markets (2009). Submitted for publication, SIAM J. Control and Optimization.
7. Dielmann, K., van der Velden, A.: Virtual power plants (vpp) - a new perspective for energy generation? In: Modern Techniques and Technologies, 2003. MTT 2003. Proceedings of the 9th International Scientific and Practical Conference of Students, Post-graduates and Young Scientists, pp. 18 – 20 (2003)
8. Elliott, R.J.: Stochastic Calculus and Applications. SpringerVerlag (1982)
9. Faghih, A., Roozbehani, M., Dahleh, M.A.: On the economic value and price-responsiveness of ramp-constrained storage. submitted, 2013, available at http://arxiv.org/abs/1211.1696 (2013)
10. Fleming, W., Soner, H.M.: Controlled Markov Processes and Viscosity Solutions, 2nd edn. Springer (2005)
11. Gross, D., Shortle, J.F., Thompson, J.M., Harris, C.M.: Fundamentals of queueing theory, 4th edn. John Wiley & Sons, Hoboken, NJ (2009)
12. Hittinger, E., Whitacre, J.F., Apt, J.: Compensating for wind variability using co-located natural gas generation and energy storage. Energy Systems **1**(4), 417–439 (2010)
13. Lamont, A.D.: Assessing the economic value and optimal structure of large-scale electricity storage. Power Systems, IEEE Trans. on **PP**(99), 1 (2012)
14. Lhndorf, N., Minner, S.: Optimal day-ahead trading and storage of renewable energies–an approximate dynamic programming approach. Energy Systems **1**(1), 61–77 (2010)
15. Meyer, R.R., Rothkopf, M.H., Smith, S.A.: Reliability and inventory in a production-storage system. Management Science **25**(8), 799–807 (1979)
16. Meyn, S.: Control Techniques for Complex Networks, 1st edn. Cambridge University Press, New York, NY, USA (2007)
17. Mitra, J.: Reliability-based sizing of backup storage. Power Systems, IEEE Trans. on **25**(2), 1198 –1199 (2010)
18. Mitra, J., Vallem, M.: Determination of storage required to meet reliability guarantees on island-capable microgrids with intermittent sources. Power Systems, IEEE Trans. on **27**(4), 2360 –2367 (2012)
19. Patton, A., Stasinos, A.: Variance and approximate confidence limits on LOLP for a single-area system. Power Apparatus and Systems, IEEE Trans. on **94**(4), 1326 – 1336 (1975)
20. Peng, S.: A generalized dynamic programming principle and hamilton-jacobi-bellman equation. Stochastics: An International Journal of Probability and Stochastic Processes **38**(2), 119–134 (1992)
21. Rebennack, S., Pardalos, P., Pereira, M.V.F., Iliadis, N.A., (Editors): Handbook of Power Systems II. Energy Systems. Springer (2012)
22. Su, H.I., El Gamal, A.: Modeling and analysis of the role of fast-response energy storage in the smart grid. In: 49th Allerton Conference on Communication, Control, and Computing (2011)
23. Taylor, J.A., Callaway, D.S., Poolla, K.: Competitive energy storage in the presence of renewables. Power Systems, IEEE Trans. on **PP**(99), 1 (2012)
24. Wang, P., Gao, Z., Bertling, L.: Operational adequacy studies of power systems with wind farms and energy storages. Power Systems, IEEE Trans. on **27**(4), 2377 –2384 (2012)