Stability of Linear Systems With Interval Time Delays Excluding Zero

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***Abstract*—The stability of linear systems with multiple, time-invariant, independent and uncertain delays is investigated. Each delay is assumed to reside within a known interval excluding zero. A delay-free sufficient comparison system is formed by replacing the delay elements with parameter-dependent filters, satisfying certain properties. It is shown that robust stability of this finite dimensional parameter-dependent comparison system, guarantees stability of the original time-delay system. This result is novel in the sense that it does not require any *a priori* knowledge regarding stability of the time-delay system for some fixed delay. When the parameter-dependent filters are formed in a particular manner using Padé approximations, an upper bound on the degree-of-conservatism of the comparison system may be obtained, which is independent of the time-delay system considered. With this, it is shown that the conservatism of this *comparison system* may be made arbitrarily small. A linear matrix ineqaulity (LMI) formulation is presented for analysis of the stability of the parameter-dependent comparison system. In the single-delay case, an eigenvalue criterion is also available for stability analysis which incurs no additional conservatism.**

***Index Terms*—Linear matrix inequality (LMI), robust stability, time delay.**

I. Introduction

A. System With Interval Delay

In this paper, we are concerned with the development of criteria for determination of the stability of the linear time-delay system (LTDS)

$$\Sigma_d : \dot{x}(t) = Ax(t) + \sum_{k=1}^{N} A_k x(t-\tau_k). \qquad (1)$$

Many investigators have examined the stability of LTDS with much of the effort directed at obtaining delay-independent and delay-dependent stability conditions (see [6] and [21]). The first of these are developed to assess stability when the unknown delays satisfy $0 \le \tau_k$, while the latter are to be employed for determining intervals of stability when the system is not delay-independent stable [21]. Often in the literature, the term delay-dependent is used interchangeably for an analysis of the first delay interval, $0 \le \tau_k \le \overline{\tau}_k$ [6]. Most articles that provide delay-dependent stability conditions use this term in this latter, more narrow sense. Herein, we will continue to employ the term in this manner. In both the delay-dependent and delay-independent analysis problems, the delay may be zero and thus the stability of the delay-free system $A + \sum_{k=1}^{N} A_k$ is a necessary condition, which may be used in development of the analysis criteria. However, for many systems of practical importance, the delay is known to be in an interval $[\underline{\tau_k}, \overline{\tau_k}]$ with $\underline{\tau_k} > 0$. Examples include thermoacoustic instability in combustion systems [1], [24] and chatter instability in machining [30]. Also, see [3], [20], [21] for a variety of systems influenced by the stabilizing effect of delay. In this paper we consider the stability analysis of (1) with $\tau_k \in [\underline{\tau_k}, \overline{\tau_k}]$ where $0 < \underline{\tau_k} < \overline{\tau_k} < \infty$, $k = 1, \ldots, N$. We will refer to this problem as the *interval-delay problem*. The distinction between the interval-delay and delay-dependent cases is critical as can be easily seen by considering the single-delay case ($N = 1$). It is well known that a LTDS may be stable for $\tau \in [\underline{\tau}, \overline{\tau}]$ but not for some $\tilde{\tau} < \underline{\tau}$. In fact, there may be multiple "pockets" of stability in τ space (see [18], [21], and [22]).

Little has appeared on this analysis problem in the literature. A result for the single-delay case is provided by [16], which establishes a necessary and sufficient stability criterion for $\tau \in [\underline{\tau}, \overline{\tau}]$, only if it is known *a priori* that the system is stable for some fixed delay $\tau_0 \in [\underline{\tau}, \overline{\tau}]$. This method is based on the Nyquist criterion and requires a sweep over frequency to guarantee stability. A similar result for the multiple interval-delay problem was provided by Huang and Zhou [15]. This result, based on the μ-framework, required *a priori* knowledge of the stability of the system for $\underline{\tau_k}$. Furthermore, the result may be very conservative since the residual uncertain delays $\tau_k - \underline{\tau_k}$ are covered rather crudely [33], [34].

To employ the result of either [16] or [15], the stability of the time-delay system must be established for a fixed delay within the interval by a separate analysis condition. This essentially provides a nominal stability foundation upon which the robust stability results of [15], [16] may be constructed. Several observations should be made regarding this nominal stability predicate and its use in conjunction with an interval condition. First, any method that may be employed for checking it could also be used for examining stability over an interval via a sweep of the delay value in the nominal condition. Second, it may be quite awkward to extend such a two-part analysis framework to controller synthesis, as it may be difficult to elegantly combine in a common framework the means for checking stability of the nominal infinite dimensional system and the means for checking preservation of stability over the interval. Finally, a computationally efficient, necessary and sufficient condition

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is not available for verifying the stability of the nominal infinite dimensional system. The result of [22] could be considered for this analysis problem. Here, a necessary and sufficient analysis technique was provided for the single delay case. This result, based on the counting of imaginary axis crossings with increasing delay, may be used to establish intervals of stability for small problems with a single delay. (An interesting earlier result along these lines was presented in [29].) However, the nature of the technique (i.e. counting crossings, symbolic analysis) prevents it from being effectively employed for synthesis (or $\mathcal{H}_\infty$ performance analysis).

The matrix pencil approach examined in [5] and [21] (Section IV-D) may be used to determine stability of systems with commensurate delays where the delay interval excludes zero. However, this is achieved in much the same fashion as in [22], in the sense that stability for such an interval is determined by starting with analysis of the delay-free system and then monitoring imaginary axis crossings of the poles, both into and out of the right half plane, as the delay value is increased. As recognized in [21], this approach cannot be extended to the non-commensurate delay case. Like [22], the approach is not amenable to further development for tackling other problems of interest such as synthesis or $\mathcal{H}_\infty$ analysis.

An alternative means for establishing the stability of the nominal infinite dimensional system is the discretized Lyapunov functional approach of Gu [10], [12]. While this method has performed well in determining stability intervals for example problems via its repeated application in a bisection, no bound exists on its conservatism. In this paper, we present an alternative approach to this stability analysis problem that eschews both the two-part analysis formulation of [15], [16] and sweep approach to interval stability of [10].

Finally, we draw the reader's attention to numerical approaches to evaluating the stability of delay-differential equations. Breda *et al.* [4] presented an approach in which the infinitesimal generator of the associated semigroup is discretized to construct a large sparse matrix, the eigenvalues of which are shown to converge to the roots of the system's characteristic quasi-polynomial as the discretization mesh is refined. In [7], a numerical algorithm involving subspace iteration on the solution of the delay-differential equation was obtained by time integration so as to form a large sparse matrix; the eigenvalues of this are then used to calculate the rightmost roots of the characteristic equation. Using an LMS method, the stability of steady state solutions are numerically evaluated in the DDE-BIFTOOL software package [8]. These approaches share two attributes that recommend their application to problems with known delay: 1) they permit the examination of the stability of LTDS with multiple (noncommensurate) delays; and 2) they provide information not only upon stability, but also on root location. For the uncertain delay problem, however, these approaches are less advantageous than for the known delay case. While they may, in principle, be employed with a sweep of the delay value(s) so as to determine intervals of stability, sweeping through the parameter space for a multiple delay case would be both computationally costly and somewhat unconvincing. We note that numerical approaches of this type cannot be extended to controller synthesis.

The contributions of this paper are threefold. First, it presents a general approach for developing finite-dimensional, parameter-dependent comparison systems for interval time-delay systems. An affirmative result in the stability analysis of the former guarantees stability of the latter. Thus, a stability criterion may be formulated without assuming any *a priori* knowledge regarding the stability of the LTDS for some fixed delay within the interval. From a viewpoint of robust control, this result is novel in the sense that robust stability of the feedback interconnection is established without requiring either open or closed loop nominal stability in contrast with small gain and μ results. The second contribution is the systematic construction of particular forms of comparison systems using Padé approximations of e^{-s}. While the *ad-hoc* engineering practice of replacing a *known* delay with its Padé approximation can yield completely erroneous results [28], [31], this paper offers an entirely rigorous criterion for the unknown delay problem. Finally, the third contribution concerns the conservatism introduced when the parameter-dependent comparison system is employed for analysis of the LTDS. When obtaining stability analysis criteria for LTDS, significant conservatism is often introduced in "removing" the infinite-dimensional nature of the delay. This conservatism appears due to both additional dynamics [9], [11], [17] and value-set covering [34]. Here, it is shown that if the parameter-dependent filters are developed in the prescribed fashion from the Padé approximation of e^{-s}, the resulting parameter-dependent comparison system has a bounded degree-of-conservatism which is known *a priori* and is independent of the system data.

B. Notation

Let $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) be the set of all real (complex) $n \times m$ matrices, and $\mathbb{C}_+$ ($\overline{\mathbb{C}}_+$) be the open (closed) right-half complex plane. For matrices $M = (m_{ij}) \in \mathbb{R}^{n_1\times n_1}$ and $N \in \mathbb{R}^{n_2\times n_2}$, $M \otimes N$ denotes the Kronecker product and $M \oplus N$ denotes the Kronecker sum. $\lambda^+_{\max}(M)$ is the maximum positive real eigenvalue of M and $\lambda^+_{\max}(M) \to 0^+$ when M has no positive real eigenvalues. In the multiple-delay case, we use τ to denote the delay vector $[\tau_1 \ldots \tau_N]$ and T to denote the delay vector set $\prod_{k=1}^{N}[\underline{\tau_k}, \overline{\tau_k}] := \{[\tau_1 \ldots \tau_N] | \tau_k \in [\underline{\tau_k}, \overline{\tau_k}]\}$. Whenever $N = 1$, we will drop the index number "1." For instance, rather than write $\tau_1 \in [\underline{\tau_1}, \overline{\tau_1}]$, we will write $\tau \in [\underline{\tau}, \overline{\tau}]$. In this case, we will denote A_1 by A_d to maintain consistency with the literature.

C. Preliminaries

The characteristic quasi-polynomial for Σ_d is given by

$$\psi_d(\tau, s) = \det\left(sI - A - \sum_{k=1}^{N} A_k e^{-\tau_k s}\right). \tag{2}$$

In the single-delay case ($N = 1$), we will rewrite (2) as

$$\psi_d(\tau, s) = \det(sI - A - A_d e^{-\tau s}).$$

To minimize the computation required, we will decompose $A_k = H_k F_k$ where $H_k \in \mathbb{R}^{n\times q_k}$ and $F_k \in \mathbb{R}^{q_k\times n}$ have full rank. Define $H_d = [H_1 \ldots H_N]$, $F_d = [F_1^T \ldots F_N^T]^T$ and $E_d(\tau, s) = \mathrm{diag}\{e^{-\tau_1 s} I_{q_1}, \ldots, e^{-\tau_N s} I_{q_N}\}$. Then, we have

$$\psi_d(\tau, s) = \det\left(sI - A - H_d E_d(\tau, s) F_d\right).$$

If $N=1$, we denote $A_d = H_d F_d$, with $H_d \equiv H_1 \in \mathbb{R}^{n\times q}$ and $F_d \equiv F_1 \in \mathbb{R}^{q\times n}$.

Definition: The spectral limit of the function $\psi_d(\tau,s)$ is

$$\sigma_d(\tau) := \sup\left(\mathrm{Re}(s) | \psi_d(\tau,s)=0\right).$$

II. Sufficient Stability Criterion

A. Filter Properties

Consider a parameter-dependent, rational transfer function $h_o(\theta,s)$, with θ belonging to Θ, a bounded set of real numbers, and $h_o(\theta,s)$ having the following properties.

$P_O - 1$) The denominator of $h_o(\theta,s)$ is Hurwitz for all $\theta\in\Theta$.

$P_O - 2$) The value set of $h_o(\theta,s)$, $\theta\in\Theta$, covers that of $e^{-\tau s}, \tau\in[\underline{\tau},\overline{\tau}]$, i.e.,

$$\begin{aligned}&\Omega_d(\omega)\subseteq\Omega_o(\omega) \qquad \forall\omega\geq 0, \qquad \text{where}\\ &\Omega_o(\omega)=\{c\in\mathbb{C}|c=h_o(\theta,j\omega),\ \theta\in\Theta\}\\ &\Omega_d(\omega)=\left\{c\in\mathbb{C}|c=e^{-j\tau\omega}, \qquad \tau\in[\underline{\tau},\overline{\tau}]\right\}.\end{aligned}$$

$P_O - 3$) There exists a $\widetilde{\theta}\in\Theta$ such that $h_o(\widetilde{\theta},j\omega)\in\Omega_d(\omega), \forall\omega\geq 0$.

Remark 1: Property P_O-2 is the conventional value set covering employed in much of the LTDS literature, although it is often hidden within the Lyapunov framework [34]. While covering the value set of the delay element has been traditionally employed to develop sufficient delay-dependent stability criteria for LTDS (see, e.g., [33]), it appears not to be sufficient for developing stability criteria for the interval-delay problem as the nominal stability necessary to obtain robust stability is not available *a priori*. As we will see in the sequel, Property P_O-2, together with Property P_O-3 may be joined into a more subtle property that we introduce as *strong covering*.

Definition 2: The parameter-dependent filter $h_o(\theta,s)$ is said to *strongly cover* $e^{-\tau s}$, $\tau\in[\underline{\tau},\overline{\tau}]$, if $\exists\,\widetilde{\theta}\in\Theta$ s.t.

$$e^{-\tau(1-\lambda)j\omega}h_o(\widetilde{\theta},\lambda j\omega)\in\Omega_o(\omega) \qquad \forall\omega\geq 0, \lambda\in[0,1],\ \tau\in[\underline{\tau},\overline{\tau}].$$

Lemma 1 (Strong Covering): Suppose $h_o(\theta,s)$ satisfies Properties P_O-2 and P_O-3. Then it strongly covers $e^{-\tau s}$. That is, for any pair $\{\tau,\lambda\}\in[\underline{\tau},\overline{\tau}]\times[0,1]$ and $\omega\geq 0$, there exists a $\theta_p\in\Theta$ such that $h_o(\theta_p,j\omega)=e^{-j\tau(1-\lambda)\omega}h_o(\widetilde{\theta},\lambda j\omega)$.

Proof: From Property P_O-3, there exists a $\widetilde{\theta}\in\Theta$ such that for $\widehat{\tau}\in[\underline{\tau},\overline{\tau}]$ and $\lambda\omega\geq 0$ we have $h_o(\widetilde{\theta},\lambda j\omega)=e^{-\widehat{\tau}\lambda j\omega}$. Therefore

$$e^{-\tau(1-\lambda)j\omega}h_o(\widetilde{\theta},\lambda j\omega)=e^{-\widetilde{\tau}j\omega} \quad \widetilde{\tau}=(1-\lambda)\tau+\lambda\widehat{\tau}\in[\underline{\tau},\overline{\tau}].$$

From Property P_O-2, there exists a $\theta_p\in\Theta$ such that $h_o(\theta_p,j\omega)=e^{-j\widetilde{\tau}\omega}$ which yields the desired result. ■

Remark 2: The strong covering property provides a homotopy from the parameter-dependent filter to the delay element along which the value set of each homotopy function remains within that of the parameter-dependent filter. This homotopy allows us to carry the stability property from a finite-dimensional *comparison system* to the LTDS.

B. Sufficient (Outer) Comparison System

Here, we introduce the parameter-dependent linear system $\Sigma_o(\theta)$

$$\begin{aligned}\dot{x}(t)&=Ax(t)+H_d u(t)\\ \dot{x}_o(t)&=A_o(\theta)x_o(t)+B_o(\theta)F_d x(t), \qquad \theta\in\Theta\\ u(t)&=C_o(\theta)x_o(t)+D_oF_dx(t)\end{aligned}\tag{3}$$

where $\left[\begin{array}{c|c}A_o(\theta)&B_o(\theta)\\\hline C_o(\theta)&D_o\end{array}\right]$ is a minimal realization of $\{h_o(\theta,s)I_q\}$. Since $A_o(\theta)$ is Hurwitz, this system is stable if and only if all the roots of the meromorphic function

$$\psi_o(\theta,s)=\det\left(sI-A-H_dh_o(\theta,s)F_d\right)$$

are in $\mathbb{C}_-$. Thus, the following two statements are equivalent.

i) $\sigma_o(\theta):=\sigma(\psi_o(\theta,s))<0$ for all $\theta\in\Theta$.

ii) Σ_o is stable for all $\theta\in\Theta$.

We will show in the sequel that robust stability of Σ_o will guarantee the robust stability of Σ_d.

C. Main Result: Sufficient Condition

Define the quasi-polynomial

$$\psi_{do}(\lambda,s):=\det\left(sI-A-H_de^{-\tau(1-\lambda)s}h_o(\widetilde{\theta},\lambda s)F_d\right) \qquad \lambda\in[0,1] \tag{4}$$

and the spectral limit function

$$\sigma_{do}(\lambda):=\sigma\left(\psi_{do}(\lambda,s)\right).\tag{5}$$

To simplify notation, the dependence of these functions on τ is suppressed.

Lemma 2: If $\sigma_d(\tau)>0$ for some $\tau\in[\underline{\tau},\overline{\tau}]$ and $\sigma_o(\widetilde{\theta})<0$, then there exists a $\lambda^*\in(0,1)$ and a $\omega^*\geq 0$ such that $\sigma_{do}(\lambda^*)=0$ and $\psi_{do}(\lambda^*,j\omega^*)=0$.

Proof: See Appendix I. ■

Theorem 1: **Main Result, Single-Delay:** If the uncertain system Σ_o is asymptotically stable for all $\theta\in\Theta$, then Σ_d will be asymptotically stable for all $\tau\in[\underline{\tau},\overline{\tau}]$.

Proof: The proof proceeds by contradiction. Toward this end, we assume that Σ_o is asymptotically stable for all $\theta\in\Theta$ and that Σ_d is unstable for some $\tau\in[\underline{\tau},\overline{\tau}]$, that is $\sigma_d(\tau)\geq 0$. We will consider first the case where $\sigma_d(\tau)>0$. Since Σ_o is asymptotically stable for all $\theta\in\Theta$, we have $\sigma_o(\theta)<0$ for all

$\theta \in \Theta$, and particularly for $\widetilde{\theta}$. Then, from Lemma 2, there exists a $\lambda^* \in (0,1)$ and an $\omega^* \geq 0$ such that

$$\begin{aligned}&\psi_{do}(\lambda^*, j\omega^*)\\&= \det\left(j\omega^* I - A - A_d e^{-\tau(1-\lambda^*)j\omega^*} h_o(\widetilde{\theta}, \lambda^* j\omega^*)\right)\\&= 0.\end{aligned}$$

From Lemma 1, there exists a $\theta_p \in \Theta$, such that $h_o(\theta_p, j\omega^*) = e^{\tau(1-\tau^*)j\omega^*} h_o(\widetilde{\theta}, \lambda^* j\omega^*)$ and, therefore

$$\begin{aligned}\psi_{do}(\lambda^*, j\omega^*) &= \det\left(j\omega^* I - A - A_d h_o(\theta_p, j\omega^*)\right)\\&\equiv \psi_o(\theta_p, j\omega^*) = 0.\end{aligned}$$

However, this implies that $\sigma_o(\theta_p) \geq 0$ which is a contradiction. Now, we consider the case where $\sigma_d(\tau) = 0$. In this instance, there exists an $\omega^* \geq 0$ such that

$$\psi_d(\tau, j\omega^*) = \det\left(j\omega^* I - A - A_d e^{-\tau j\omega^*}\right) = 0.$$

From Property $P_O - 2$, there exists a $\widehat{\theta}$, $\widehat{\theta} \in \Theta$, such that $h_o(\widehat{\theta}, j\omega^*) = e^{-\tau j\omega^*}$ and, therefore

$$\psi_o(\widehat{\theta}, j\omega^*) = \det\left(j\omega^* I - A - A_d h_o(\widehat{\theta}, j\omega^*)\right) = 0.$$

However, this implies $\sigma_o(\widehat{\theta}) \geq 0$ which contradicts our assumption regarding the asymptotic stability of Σ_o. ∎

To present the corresponding multiple-delay results, we now introduce the system $\Sigma_o^m(\theta)$ as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + H_d u(t)\\\dot{x}_o(t) &= A_o(\theta)x_o(t) + B_o(\theta)F_d x(t), \qquad \theta \in \Theta\\u(t) &= C_o(\theta)x_o(t) + D_o F_d x(t)\end{aligned} \tag{6}$$

where $\theta := [\theta_1 \ldots \theta_N]$ is the vector of parameters and $\Theta = \prod_{k=1}^N \Theta_k := \{[\theta_1 \ldots \theta_N] | \theta_k \in \Theta_k\}$, where each Θ_k is a bounded set of real numbers. (Note that θ in this case is a vector of real parameters). Here, $\left[\begin{array}{c|c} A_o(\theta) & B_o(\theta) \\ \hline C_o(\theta) & D_o \end{array}\right]$ is a minimal realization of

$$\{M_o(\theta, s)\} := \operatorname{diag}\left\{h_{o_1}(\theta_1, s)I_{q_1}, \ldots, h_{o_N}(\theta_N, s)I_{q_N}\right\} \tag{7}$$

where each $h_{o_k}(\theta_k, s)$ satisfies properties $P_O - 1$ through $P_O - 3$.

Theorem 2: **Main Result, Multiple-Delay:** If the uncertain system Σ_o^m is asymptotically stable for all $\theta \in \Theta$, then Σ_d will be asymptotically stable for all $\tau \in T$.

Proof: Similar to Theorem 1; see [25]. ∎

III. Necessary Stability Criterion

A. Filter Properties

Consider a parameter-dependent, rational transfer function $h_i(\theta, s)$ having the following properties.

$P_I - 1$) The denominator of $h_i(\theta, s)$ is Hurwitz for $\theta \in \Theta$.

$P_I - 2$) The value set of $e^{-\tau s}$, $\tau \in [\underline{\tau}, \overline{\tau}]$, covers that of $h_i(\theta, s)$, $\theta \in \Theta$, i.e.,

$$\begin{aligned}&\Omega_i(\omega) \subseteq \Omega_d(\omega), \forall \omega \geq 0, \text{ where}\\&\Omega_i(\omega) := \{c \in \mathbb{C} | c = h_i(\theta, j\omega), \theta \in \Theta\}.\end{aligned}$$

We will show that such an inner filter results in a necessary condition for stability of Σ_d.

Lemma 3: Suppose $h_i(\theta, s)$ satisfies Properties $P_I - 1$ and $P_I - 2$. For any triplet $\{\theta, \lambda, \tau\} \in \Theta \times [0,1] \times [\underline{\tau}, \overline{\tau}]$ and $\omega \geq 0$, there exists a $\tau_p \in [\underline{\tau}, \overline{\tau}]$ such that $e^{-\tau_p j\omega} = e^{-\tau(1-\lambda)j\omega} h_i(\theta, \lambda j\omega)$.

Proof: From Property $P_I - 2$, for any $\theta \in \Theta$, $\lambda \in [0,1]$, and $\omega \geq 0$, there exists a $\widetilde{\tau}$ such that $e^{-\widetilde{\tau}\lambda j\omega} = h_i(\theta, \lambda j\omega)$. Multiplying by $e^{-\tau(1-\lambda)j\omega}$, we have

$$e^{-\tau(1-\lambda)j\omega} h_i(\theta, \lambda j\omega) = e^{-\tau_p j\omega}, \ \tau_p = (1-\lambda)\tau + \lambda\widetilde{\tau} \in [\underline{\tau}, \overline{\tau}]$$

which is the desired result. ∎

B. Necessary (Inner) Comparison System

Herein, we introduce the parameter-dependent system Σ_i

$$\begin{aligned}\dot{x}(t) &= Ax(t) + H_d u(t)\\\dot{x}_i(t) &= A_i(\theta)x_i(t) + B_i(\theta)F_d x(t), \qquad \theta \in \Theta\\u(t) &= C_i(\theta)x_i(t) + D_i(\theta)F_d x(t)\end{aligned} \tag{8}$$

where $\left[\begin{array}{c|c} A_i(\theta) & B_i(\theta) \\ \hline C_i(\theta) & D_i \end{array}\right]$ is a minimal realization of $\{h_i(\theta, s)I_q\}$. Since $A_i(\theta)$ is Hurwitz, Σ_i is stable if and only if all the roots of the meromorphic function

$$\psi_i(\theta, s) = \det\left(sI - A - H_d h_i(\theta, s)F_d\right)$$

are in $\mathbb{C}_-$. Therefore, $\sigma_i(\theta) := \sigma(\psi_i(\theta, s)) < 0$ for all $\theta \in \Theta$, if and only if Σ_i is stable for all $\theta \in \Theta$. We will demonstrate that stability of Σ_i is a necessary condition for stability of Σ_d.

C. Necessary Condition

Define the quasi-polynomial

$$\psi_{di}(\lambda, s) := \det\left(sI - A - H_d e^{-\tau(1-\lambda)s} h_i(\theta, \lambda s)F_d\right), \ \lambda \in [0,1] \tag{9}$$

and the spectral limit function

$$\sigma_{di}(\lambda) := \sigma\left(\psi_{di}(\lambda, s)\right). \tag{10}$$

(The dependence of these functions on θ and τ is suppressed to simplify notation.)

Lemma 4: If $\sigma_d(\tau) < 0$ for all $\tau \in [\underline{\tau}, \overline{\tau}]$ and $\sigma_i(\theta) > 0$ for some $\theta \in \Theta$, then there exists a $\lambda^* \in (0,1)$ and a $\omega^* \geq 0$ such that $\sigma_{di}(\lambda^*) = 0$ and $\psi_{di}(\lambda^*, j\omega^*) = 0$.

Proof: Follows in the same fashion as Lemma 2. ■

Theorem 3: **Necessary Condition, Single-Delay:** If the uncertain linear time-delay system Σ_d is asymptotically stable for all $\tau \in [\underline{\tau}, \overline{\tau}]$, then Σ_i is asymptotically stable for all $\theta \in \Theta$.

Proof: The proof proceeds by contradiction. Toward this end, we assume that Σ_d is asymptotically stable for all τ, $\tau \in [\underline{\tau}, \overline{\tau}]$, and that Σ_i is unstable for some $\theta \in \Theta$, that is $\sigma_i(\theta) \geq 0$. First, we consider the case where $\sigma_i(\theta) > 0$ and then will return to the case $\sigma_i(\theta) = 0$. Since Σ_d is asymptotically stable for all $\tau \in [\underline{\tau}, \overline{\tau}]$, we have $\sigma_d(\tau) < 0$ for all $\tau \in [\underline{\tau}, \overline{\tau}]$. We may pick any $\tau \in [\underline{\tau}, \overline{\tau}]$ and define $\psi_{di}(\lambda, s)$ using these choices for τ and θ. From Lemma 4, there exists a $\lambda^* \in (0,1)$ and a $\omega^* \geq 0$ such that

$$\begin{aligned}
&\psi_{di}(\lambda^*, j\omega^*) \\
&\quad = \det\left(j\omega^* I - A - A_d e^{-\tau(1-\lambda^*)j\omega^*} h_i(\theta, \lambda^* j\omega^*)\right) \\
&\quad = 0.
\end{aligned}$$

From Lemma 3, there exists a $\tau_p \in [\underline{\tau}, \overline{\tau}]$, such that $e^{-\tau_p j\omega^*} = e^{-\tau(1-\lambda^*)j\omega^*} h_i(\theta, \lambda^* j\omega^*)$ and, therefore

$$\begin{aligned}
\psi_{di}(\lambda^*, j\omega^*) &= \det\left(j\omega^* I - A - A_d e^{-j\tau_p \omega^*}\right) \\
&= \psi_d(\tau_p, j\omega^*) = 0.
\end{aligned}$$

However, this implies that $\sigma_d(\tau_p) \geq 0$ which is a contradiction. Now, we consider the case where $\sigma_i(\theta) = 0$. In this instance, there exists an $\omega^* \geq 0$ such that

$$\psi_i(\theta, j\omega^*) = \det\left(j\omega^* I - A - A_d h_i(\theta, j\omega^*)\right) = 0.$$

Now, from Property $P_I - 2$ of $h_i(\theta, s)$, there exists a $\hat{\tau} \in [\underline{\tau}, \overline{\tau}]$, such that $e^{-\hat{\tau} j\omega^*} = h_i(\theta, j\omega^*)$ and, therefore

$$\psi_d(\hat{\tau}, j\omega^*) = \det\left(j\omega^* I - A - A_d e^{-\hat{\tau} j\omega^*}\right) = 0.$$

However, this implies $\sigma_d(\hat{\tau}) \geq 0$ which contradicts our assumption regarding the asymptotic stability of Σ_d. ■

To present the corresponding multiple-delay results, we now introduce the system $\Sigma_i^m(\theta)$ as

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + H_d u(t) \\
\dot{x}_i(t) &= A_i(\theta)x_i(t) + B_i(\theta)F_d x(t), \qquad \theta \in \Theta \\
u(t) &= C_i(\theta)x_i(t) + D_i(\theta)F_d x(t)
\end{aligned} \tag{11}$$

where $\left[\begin{array}{c|c} A_i(\theta) & B_i(\theta) \\ \hline C_i(\theta) & D_i \end{array}\right]$ is a minimal realization of

$$\{M_i(\theta, s)\} := \operatorname{diag}\left\{h_{i_1}(\theta_1, s)I_{q_1}, \ldots, h_{i_N}(\theta_N, s)I_{q_N}\right\}$$

where each $h_{i_k}(\theta_k, s)$ satisfies properties $P_I - 1$ and $P_I - 2$.

Theorem 4: **Necessary Condition, Multiple-Delay:** If the uncertain linear time-delay system Σ_d is asymptotically stable for all $\tau \in T$, then Σ_i^m is asymptotically stable for all $\theta \in \Theta$.

Proof: Similar to Theorem 3; see [25]. ■

IV. Candidate Filters and Their Conservatism

A. Preliminaries

We now turn our attention to the development of filters $h_o(\theta, s)$ and $h_i(\theta, s)$ such that $P_O - 1$ through $P_O - 3$ and $P_I - 1$, $P_I - 2$ hold. We seek to develop these functions such that the degree-of-conservatism of the analysis criterion can both be bounded and made arbitrarily small.

Definition 3: Given a continuous function $g(q) : [0, \infty) \to \mathcal{D}$ where $\mathcal{D} = \{z \in \mathbb{C} |\, |z| = 1\}$, letting Γ_r be the path created by mapping the interval $q \in [0, r]$ via $g(q)$ to $\mathcal{D}$, we define a *continuous argument* (phase) function for the value $g(r)$ as $\operatorname{Arg}(g(r)) = \arg(g(r)) + 2\pi n(\Gamma_r, 0)$ where $\arg(z) \in (-2\pi, 0]$ is the unique argument of $z \in \mathbb{C}$, $z \neq 0$, and $n(\Gamma_r, a)$ is the winding number[1] of path Γ about a.

Lemma 5: [32] **Properties of the Padé Argument:** Let $p_l(s)$ be the l^{th} order diagonal Padé approximation to e^{-s}, with $l \geq 3$. Define $\phi_p(\omega) := \operatorname{Arg}(p_l(j\omega))$. Then, by definition $\phi_p(\omega)$ is a continuous function of ω with $\phi_p(0) = 0$. Moreover

$$\frac{d^2\phi_p}{d\omega^2}(\omega) > 0, \qquad \omega > 0 \tag{12}$$

$$\frac{d}{d\omega}\phi_p(\omega) = -\frac{T_l(\omega)}{\omega^{2l} + T_l(\omega)}, \qquad \omega \geq 0 \tag{13}$$

where

$$T_l(\omega) = \sum_{k=0}^{l-1} a_{(l,k)} \omega^{2k} \tag{14}$$

and

$$a_{(l,k)} = \frac{[2(l-k)]!(2l-k)!}{k!\,[(l-k)!]^2} > 0. \tag{15}$$

The following properties follow from Lemma 5 and are particularly useful in generating the parameter-dependent filters:

$$\frac{d\phi_p}{d\omega}(0) = -1 \tag{16}$$

$$0 > \frac{d\phi_p}{d\omega}(\omega) \geq -1, \qquad \omega \in \mathbb{R} \tag{17}$$

$$-y \leq \phi_p(y), \qquad y \geq 0. \tag{18}$$

[1]For clockwise paths winding numbers are negative. See [23] for more details regarding winding numbers.

Lemma 6: [32] $\forall \omega \in \mathbb{R}, |p_l(j\omega)| = 1$.

Equation (17) provides an upper bound on $|(d/d\omega)\mathrm{Arg}(p_l(j\omega))|$, which is very loose at high frequencies. The following Lemma, provides a much tighter bound on $|(d/d\omega)\mathrm{Arg}(p_l(j\omega))|$ for high frequencies. We will exploit the bound given in (19).

Lemma 7: For each Padé order l, there exists a constant $L > 0$, such that

$$\forall \omega \in \mathbb{R} \quad \left|\frac{d}{d\omega}\mathrm{Arg}\left(p_l(j\omega)\right)\right| \leq \min\left(1, \frac{L}{\omega^2}\right). \tag{19}$$

Moreover, $4(l^2+1)$ is an upper bound for L.

Proof: See Appendix B. ∎

Lemma 8: [19] Let $p_l(s)$ be the l^{th} order diagonal Padé approximation to e^{-s}, $l \geq 1$. For every $\omega \geq 0$, we have

$$\left|e^{-j\omega} - p_l(j\omega)\right| \leq 2\left(\frac{\omega e}{4l+2}\right)^{2l+1}$$

Equality holds for $\omega = 0$ only.

B. Outer Parameter-Dependent Filter

1) Definition: Let us denote

$$\tau_m = \frac{\overline{\tau} + \underline{\tau}}{2} \quad b = \frac{\overline{\tau} - \underline{\tau}}{2} \quad \kappa = \frac{\tau_m}{b}.$$

Consider the outer parameter-dependent filter

$$h_o(\theta, s) := p_l\left([\tau_m - \alpha_o b]s\right) p_l(2\alpha_o \theta s), \qquad \theta \in \Theta \tag{20}$$

where

$$\alpha_o := \min\left\{\alpha | 1 < \alpha < \kappa, \ \Psi_o(\alpha) = 0\right\} \tag{21}$$

$$\Psi_o(\alpha) := \mathrm{Arg}\left(p_l\left([\kappa - \alpha]j\omega_o\right) p_l(2\alpha j\omega_o)\right) - \mathrm{Arg}\left(e^{-[\kappa+1]j\omega_o}\right) \tag{22}$$

$$\omega_o := \min\left\{\omega > 0 | p_l(2j\omega) = 1\right\} \tag{23}$$

$$\Theta := (0, b]. \tag{24}$$

Note that this definition of ω_o implies that $\mathrm{Arg}(p_l(2j\omega_o)) = -2\pi$, and $p_l(j[\kappa - \alpha]\omega_o)p_l(j2\alpha\omega_o) = e^{-j[\kappa+1]\omega_o}$ for $\alpha = \alpha_o$. At points in the exposition it will be necessary to emphasize the dependence of α_o, Ψ_o, and ω_o upon the Padé order l. To do so, we will write $\alpha_o^{[l]}$, $\Psi_o^{[l]}$, and $\omega_o^{[l]}$. Whenever possible, we will suppress the superscript $[l]$ notation.

Lemma 9: For $l \geq 3$, there exists a frequency ω_o satisfying (23).

Proof: The Padé approximation has unit magnitude at all frequencies and its argument sweeps continuously from 0 to a terminal phase of $-\pi l$. The result follows directly. ∎

It can also be shown that

$$\pi < \omega_o^{[l]} < 1.24\pi \quad \lim_{l\to\infty} \omega_o^{[l]} = \pi$$

Our next result will be important in establishing $P_O - 3$.

TABLE I
MINIMUM PADÉ ORDER REQUIRED TO GUARANTEE EXISTENCE OF $\alpha_o^{[l]} < 1 + (\pi/\omega_o^{[l]})$ FOR A GIVEN κ

Range of κ	min l	Range of κ	min l
$\kappa \in [1.0008, 1.0078]$	6	$\kappa \in [5.6395, 6.8935]$	6
$\kappa \in [1.0079, 1.1118]$	5	$\kappa \in [6.8936, 7.8652]$	7
$\kappa \in [1.1119, 3.5263]$	4	$\kappa \in [7.8653, 8.6640]$	8
$\kappa \in [3.5264, 5.6394]$	5	$\kappa \in [8.6641, 9.3758]$	9

Lemma 10: For every $\kappa > 1$ there exists an integer l_o^κ such that for each $l \geq l_o^\kappa$ there exists an $\alpha_o^{[l]} < 1 + (\pi/\omega_o^{[l]})$ satisfying (21). Furthermore, the minimum integer l satisfying

$$2\arcsin\left[\left(\frac{\kappa e \omega_o^{[l]}}{2l+1}\gamma\right)^{2l+1}\right] < \frac{1}{2}\omega_o^{[l]}\left(\frac{\kappa}{\mu} - 1\right) \tag{25}$$

is an upper bound for l_o^κ, where $\mu = \max(1, (\kappa/1.8))$, and $\gamma = \max(1/\mu, (\mu-1)/2\mu)$.

Proof: See Appendix B. ∎

Remark 3: Lemma 10 was introduced only to assure existence of $\alpha_o^{[l]} < 1 + (\pi/\omega_o^{[l]})$ for sufficiently large l for each κ. However, the actual Padé order required for existence of $\alpha_o^{[l]} < 1 + (\pi/\omega_o^{[l]})$ is much less than what condition (25) provides. In practice, l and α_o may be easily determined from a numerical analysis. See Table I.

2) Satisfaction of Properties: For the remainder of this section, $h_o(\theta, s)$ is as specified in (20) with Padé order l chosen such that α_o and ω_o may be found from (21) and (23). We will demonstrate that the function $h_o(\theta, s)$ satisfies Properties $P_O - 1$ through $P_O - 3$. First, since the denominator of $p_l(s)$ is Hurwitz $h_o(\theta, s)$ will satisfy Property $P_O - 1$. The following results demonstrate that Properties $P_O - 2$ and $P_O - 3$ are also satisfied.

Lemma 11: For each $\omega \geq 0$, we have

$$\Omega_d(\omega) \subseteq \Omega_o(\omega)$$

where

$$\Omega_o(\omega) = \left\{c \in \mathbb{C} | c = h_o(\theta, j\omega), \ 0 < \theta \leq b\right\}$$
$$\Omega_d(\omega) = \left\{c \in \mathbb{C} | c = e^{-\tau j\omega}, \ \underline{\tau} \leq \tau \leq \overline{\tau}\right\}.$$

That is, for every $\tau \in [\underline{\tau}, \overline{\tau}]$ and $\omega \geq 0$, there exists a $\theta \in (0, b]$ such that $e^{-\tau j\omega} = h_o(\theta, j\omega)$.

Proof: See Appendix B. ∎

Lemma 12: Suppose l is chosen such that $\alpha_o^{[l]}$ and $\omega_o^{[l]}$ exist and $\alpha_o^{[l]} < 1 + (\pi/\omega_o^{[l]})$. There exists a $\widetilde{\theta}$ such that $h_o(\widetilde{\theta}, j\omega) \in \Omega_d(\omega)$, $\forall \omega \geq 0$.

Proof: See Appendix B. ∎

3) Convergence of α_o:

Lemma 13: Define a sequence $\{\alpha_o^{[l]}\}$, $l \geq l_o^\kappa$ from (21). Then, $\lim_{l\to\infty} \alpha_o^{[l]} = 1$. Furthermore, for all $l > l_o^\kappa$

$$\alpha_o^{[l]} - 1 < \overline{R}_o^{[l]} := 2\overline{R}_1^{[l]}\overline{R}_2^{[l]} + \overline{R}_1^{[l]} + \overline{R}_2^{[l]} \tag{26}$$

where

$$\overline{R}_1^{[l]} = \left(\frac{(\kappa-1)\pi e}{2l+1}\right)^{2l+1}, \text{ and } \overline{R}_2^{[l]} = \left(\frac{2\kappa\pi e}{2l+1}\right)^{2l+1}.$$

Proof: See Appendix B. ∎

C. Inner Parameter-Dependent Filter

1) Definition: Consider the inner parameter-dependent filter

$$h_i(\theta, s) := p_l\left(\left[\tau_m - \frac{1}{\alpha_i}b\right]s\right)p_l\left(\frac{2}{\alpha_i}\theta s\right), \qquad \theta \in \Theta \tag{27}$$

where

$$\alpha_i := \min\{\alpha | 1 < \alpha,\ \Psi_i(\alpha) = 0\} \tag{28}$$

$$\Psi_i(\alpha) := \mathrm{Arg}\left(p_l\left(\left[\kappa - \frac{1}{\alpha}\right]j\pi\right)\right) - \mathrm{Arg}\left(e^{-[\kappa-1]j\pi}\right) \tag{29}$$

and Θ is defined as before. Note that since both $p_l(j[\kappa - (1/\alpha)]\pi)$ and $e^{-j[\kappa-1]\pi}$ have unit magnitude, $p_l(j[\kappa - (1/\alpha)]\pi) = e^{-j[\kappa-1]\pi}$ for $\alpha = \alpha_i$.

Lemma 14: For every $\kappa > 1$ there exists an integer l_i^κ such that for each integer $l \geq l_i^\kappa$ there exists an α_i satisfying (28). Furthermore, the minimum integer l satisfying

$$2\arcsin\left[\left(\frac{\kappa\pi e}{2l+1}\right)^{2l+1}\right] < \frac{\pi}{2} \tag{30}$$

is an upper bound for l_i^κ.

Remark 4: The lemmas in this subsection can be proven using the same ideas as the preceding subsection for outer parameter-dependent filters. The proofs can be found in [25].

Remark 5: With l chosen such that $l \geq max(l_i^\kappa, l_o^\kappa)$, both α_o and α_i will exist. Furthermore, with the outer and inner filters, $h_o(\theta, s)$ and $h_i(\theta, s)$, chosen to be of the same order, they are related to each other through a dilation, a relationship which will be important in establishing conservatism bounds.

2) Satisfaction of Properties: Clearly, this choice of inner approximation satisfies Property $P_I - 1$. The following result indicates that Property $P_I - 2$ is also satisfied.

Lemma 15: Suppose that $h_i(\theta, s)$ is as specified in (27) with Padé order l chosen such that α_i may be found satisfying (28). For each ω, we have

$$\Omega_i(\omega) \subseteq \Omega_d(\omega), \ \text{ where}$$
$$\Omega_i(\omega) = \{c \in \mathbb{C} | c = h_i(\theta, j\omega),\ 0 < \theta \leq b\}.$$

That is, for every $\theta, 0 < \theta \leq b$, and $\omega \geq 0$, there exists a $\tau, 0 < \underline{\tau} \leq \tau \leq \overline{\tau}$, such that $h_i(\theta, j\omega) = e^{-\tau j\omega}$.

3) Convergence of α_i:

Lemma 16: Define a sequence of $\{\alpha_i^{[l]}\}$, $l \geq l_i^\kappa$ from (28). Then $\lim_{l\to\infty} \alpha_i^{[l]} = 1$. Furthermore, for all $l \geq l_i^\kappa$

$$\frac{\alpha_i^{[l]} - 1}{\alpha_i^{[l]}} < \overline{R}_i^{[l]} \tag{31}$$

$$\text{where, } \ \overline{R}_i^{[l]} \equiv \left(\frac{\kappa\pi e}{4l+2}\right)^{2l+1}. \tag{32}$$

D. Conservatism

1) Single-Delay

Definition 4: The *delay margin* δ^* for the system (1) about a mean delay value of τ_m, is defined by

$$\delta^* := \sup\{\delta < \tau_m | (1) \text{ is asymptotically stable on } [\tau_m - \delta, \tau_m + \delta]\}.$$

Definition 5: Suppose $\mathcal{C}$ is a condition that ensures that (1) is asymptotically stable on $[\tau_m - \delta_\mathcal{C}, \tau_m + \delta_\mathcal{C}]$. If (1) has a *delay margin* about τ_m of $\delta^* > 0$, then the *degree-of-conservatism (d.o.c.)* of condition $\mathcal{C}$ at τ_m is defined by

$$d.o.c._\mathcal{C}(\tau_m) := \frac{\delta^* - \delta_\mathcal{C}^*}{\delta^*}$$

where

$$\delta_\mathcal{C}^* := \sup\{\delta_\mathcal{C} | \mathcal{C} \text{ is true on } [\tau_m - \delta_\mathcal{C}, \tau_m + \delta_\mathcal{C}]\}.$$

Moreover, $\delta_\mathcal{C}^*$ is said to be the *delay margin provided by* $\mathcal{C}$ at τ_m.

Corollary 1: **Stability Employing Outer Filter:** Given a mean value of delay τ_m, form $h_o(\theta, s)$ using (20) and Σ_o as in (3). We will denote this system in the sequel as $\Sigma_o(b; \theta)$ to make the description's dependence on delay interval half-width b explicit. If

$$\Sigma_o(b; \theta) \text{ is asymptotically stable for all } 0 < \theta \leq b, \tag{$\mathcal{P}$}$$

then Σ_d is asymptotically stable for all $\tau \in [\tau_m - b, \tau_m + b]$.

Proof: The result follows directly from Theorem 1. ■

Corollary 2: If Σ_d is asymptotically stable for all $\tau \in [\tau_m - b, \tau_m + b]$, then $\Sigma_i(b; \theta)$ is asymptotically stable for all $0 < \theta \leq b$.

Proof: The result follows directly from Theorem 3. ■

Next, we show that the d.o.c. of Corollary 1 is always bounded by a function of $\alpha_o^{[l]}\alpha_i^{[l]}$.

Theorem 5: Given τ_m, the $d.o.c.$ of the *delay margin provided by* Corollary 1 at τ_m satisfies

$$\text{d.o.c.}_\mathcal{P}(\tau_m) \leq \frac{\alpha_o^{[l]}\alpha_i^{[l]} - 1}{\alpha_o^{[l]}\alpha_i^{[l]}}. \tag{33}$$

Furthermore, $\text{d.o.c.}_\mathcal{P}(\tau_m) < \overline{R}_o^{[l]} + \overline{R}_i^{[l]}$, and $\text{d.o.c.}_\mathcal{P}(\tau_m) \to 0$ as $l \to \infty$.

Proof: Let δ^* be the *delay margin* of (1) at τ_m, and suppose that $\delta^* < \tau_m$. Let δ_o^* be the *delay margin* provided by Corollary 1 with $h_o(\theta, s)$ chosen as (20) with Padé order l sufficiently large $l \geq \max(l_\kappa^o, l_\kappa^i)$. That is

$$\delta_o^* := \sup\{\delta_o | \Sigma_o(\delta_o; \theta) \text{ is asymptotically stable } \ \forall\theta \in (0, \delta_o]\}$$

Define δ_i^*

$$\delta_i^* := \sup\{\delta_i | \Sigma_i(\delta_i; \theta) \text{ is asymptotically stable } \ \forall\theta \in (0, \delta_i]\}$$

where the system $\Sigma_i(\delta_i;\theta)$ is formed as in (8) using $h_i(\theta,s)$ with the same Padé order l. Due to the dilation relationship between $h_i(\theta,s)$ and $h_o(\theta,s)$, we clearly have

$$\frac{\delta_i^*}{\alpha_i^{[l]}} = \alpha_o^{[l]}\delta_o^*$$

From Theorem 3, $\Sigma_i(\delta;\theta)$ is robustly stable for $\theta \in (0,\delta]$ whenever (1) is asymptotically stable on $[\tau_m-\delta,\tau_m+\delta]$. Also, from Theorem 1, (1) is robustly stable on $[\tau_m-\delta,\tau_m+\delta]$ whenever $\Sigma_o(\delta;\theta)$ is asymptotically stable for $\theta \in (0,\delta]$. Therefore,

$$\delta_i^* \geq \delta^* \geq \delta_o^*$$

Then, a bound on the d.o.c. follows from:

$$\text{d.o.c.}_{\mathcal{P}}(\tau_m) = \frac{\delta^*-\delta_o^*}{\delta^*} = 1-\frac{\delta_o^*}{\delta^*} \leq 1-\frac{\delta_o^*}{\delta_i^*} = \frac{\alpha_o^{[l]}\alpha_i^{[l]}-1}{\alpha_o^{[l]}\alpha_i^{[l]}}.$$

Clearly, $\lim_{l\to\infty} \text{d.o.c.}_{\mathcal{P}}(\tau_m) \to 0$ since $\alpha_o^{[l]} \to 1$ and $\alpha_i^{[l]} \to 1$ as $l \to \infty$. Note that

$$\frac{\alpha_o^{[l]}\alpha_i^{[l]}-1}{\alpha_o^{[l]}\alpha_i^{[l]}} < \frac{\alpha_o^{[l]}\alpha_i^{[l]}-1}{\alpha_i^{[l]}} = \alpha_o^{[l]}-1+\frac{\alpha_i^{[l]}-1}{\alpha_i^{[l]}}.$$

From (26) and (32), we have

$$\alpha_o^{[l]}-1 < \overline{R}_o^{[l]}, \text{ and } \frac{\alpha_i^{[l]}-1}{\alpha_i^{[l]}} < \overline{R}_i^{[l]}, \text{ and thus :}$$
$$d.o.c._{\mathcal{P}}(\tau_m) < \overline{R}_o^{[l]} + \overline{R}_i^{[l]}$$

which clearly indicates the rapid decrease in conservatism with increasing Padé order. ∎

Remark 6: The careful reader will notice that the parameters $\alpha_o^{[l]}$ and $\alpha_i^{[l]}$ are functions of κ and, hence, b, raising questions regarding how to properly interpret Theorem 5. These may be resolved by considering how δ_o^* and δ_i^* are defined. Consider δ_o^* for convenience. In principle, given τ_m, for any half-width b, we may determine κ and α_o, define the comparison system $\Sigma_o(b;\theta)$, and ascertain its stability. The margin δ_o^* is the largest b for which stability of the resulting comparison system is achieved. The value $\alpha_o^{[l]}$ to be used in Theorem 5 is the value employed in constructing the comparison system with this largest value of b. The same holds regarding the value of $\alpha_i^{[l]}$. With all other parameters fixed, both $\alpha_o^{[l]}$ and $\alpha_i^{[l]}$ will decrease with increasing b and so the potential degree of conservatism of successive tests with increasing b will decrease as well.

Lemma 17: For each $\theta \in (0,b]$,

$$\lim_{l\to\infty} \|h_o(\theta,s)-h_i(\theta,s)\|_\infty = 0.$$

Proof: See Appendix B. ∎

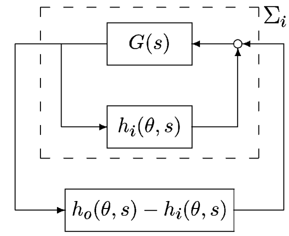

Fig. 1. Perturbation of the system Σ_i.

Theorem 6: **Asymptotic Necessity of Outer Comparison System:** Let δ^* be the *delay margin* about a mean delay value of τ_m for the (asymptotically stable) system Σ_d. Then, for any $0 < b < \delta^*$, there exists a comparison system $\Sigma_o(b,\theta)$ (developed with high enough Padé order) that proves asymptotic stability of Σ_d for all $\tau \in [\tau_m-b,\tau_m+b]$.

Proof: For any positive $b < \delta^*$, the system Σ_d is asymptotically stable for all $\tau \in [\tau_m-b,\tau_m+b]$. Corollary 2 then implies that for any $l \geq l_i^\kappa$, $\Sigma_i(b,\theta)$ is asymptotically stable for all $\theta \in (0,b]$. For any $l \geq \max(l_i^\kappa, l_o^\kappa)$, consider a perturbation of the system $\Sigma_i(b,\theta)$, as shown in Fig. 1. Employing the small gain theorem, for each $\theta \in (0,b]$, the feedback interconnection of $\Sigma_i(b,\theta)$ and $h_o(\theta,s)-h_i(\theta,s)$ is stable, provided that the following condition is satisfied:

$$\|h_o(\theta,s)-h_i(\theta,s)\|_\infty \|\Sigma_i(b,\theta)\|_\infty < 1.$$

A sufficient condition for stability of the feedback interconnection of $\Sigma_i(b,\theta)$ and $h_o(\theta,s)-h_i(\theta,s)$ for all $\theta \in (0,b]$ is then given by

$$\sup_{\theta\in(0,b]} \|h_o(\theta,s)-h_i(\theta,s)\|_\infty \sup_{\theta\in(0,b]} \|\Sigma_i(b,\theta)\|_\infty < 1. \quad (34)$$

From Lemma 17, $\|h_o(\theta,s)-h_i(\theta,s)\|_\infty$ can be made arbitrarily small by increasing the Padé order. It then follows naturally that $\sup_{\theta\in(0,b]} \|h_o(\theta,s)-h_i(\theta,s)\|_\infty$ can be made arbitrarily small too. That is, there exists an integer $L \geq \max(l_i^\kappa, l_o^\kappa)$ such that for any integer $l \geq L$, condition (34) is satisfied. Therefore, with any $l \geq L$, the system $\Sigma_o(b,\theta)$ is asymptotically stable for all $\theta \in (0,b]$. This conclusion along with Corollary 1 completes the proof. ∎

4) Multiple-Delay:

Corollary 3: **Stability Employing Outer Filter:** Given a nominal delay vector $\tau_m := [\tau_{1m}\dots\tau_{Nm}]$, form $M_o(\theta,s)$ using (7) and Σ_o^m as in (6). We shall denote this system in the sequel as $\Sigma_o^m(b;\theta)$ to make the description's dependence on the delay interval half-width $b := [b_1,\dots,b_N]$ explicit. If

$$\Sigma_o^m(b;\theta) \text{ is asymptotically stable } \quad \forall\theta\in\Theta \quad (\mathcal{P})$$

where

$$\Theta := \prod_{k=1}^{N}(0,b_k]$$

then Σ_d is asymptotically stable for all $\tau \in T$, where

$$T := \prod_{k=1}^{N} [\tau_{km} - b_k, \tau_{km} + b_k].$$

Proof: The result is a special case of Theorem 2. ■

Corollary 4: If the uncertain system Σ_d is asymptotically stable for all $\tau \in T \equiv \prod_{k=1}^{N}[\tau_{km} - b_k, \tau_{km} + b_k]$, then $\Sigma_i^m(b;\theta)$ is asymptotically stable for all $\theta \in \Theta \equiv \prod_{k=1}^{N}(0, b_k]$.

Proof: The result is a special case of Theorem 4. ■

Definition 6: The *delay margin* δ^* for (1) about a nominal delay vector τ_m with the aspect ratio vector $v := [v_1 \dots v_N]$, where $\min_{1\le k\le N}(v_k) = 1$, is defined by

$$\delta^* := \sup\left\{\delta | (1) \text{ is asymptotically stable on } \prod_{k=1}^{N} [\tau_{km} - v_k\delta, \tau_{km} + v_k\delta]\right\}.$$

Definition 7: Define an aspect ratio vector $v := [v_1, \dots, v_N]$, $\min_{1\le k\le N}(v_k) = 1$. Suppose $\mathcal{C}$ is a condition that ensures that (1) is asymptotically stable on $\prod_{k=1}^{N}[\tau_{km} - v_k\delta_{\mathcal{C}}, \tau_{km} + v_k\delta_{\mathcal{C}}]$. If (1) has a delay margin of δ^* about τ_m with aspect ratio vector v, then the d.o.c. of condition $\mathcal{C}$ at τ_m is defined by

$$\text{d.o.c.}_{\mathcal{C}}(\tau_m, v) := \frac{\delta^* - \delta^*_{\mathcal{C}}}{\delta^*}$$

where

$$\delta^*_{\mathcal{C}} := \sup\left\{\delta_{\mathcal{C}} | \mathcal{C} \text{ is true on } \prod_{k=1}^{N} [\tau_{km} - v_k\delta_{\mathcal{C}}, \tau_{km} + v_k\delta_{\mathcal{C}}]\right\}.$$

Moreover, $\delta^*_{\mathcal{C}}$ is said to be the *delay margin provided by* $\mathcal{C}$ *at* τ_m, with aspect ratio vector v. Note that the dependence of $\delta^*_{\mathcal{C}}$ on τ_m is suppressed in the notation.

Theorem 7: Given $\tau_m = [\tau_{1m}, \dots, \tau_{Nm}]$, assume that the maximum of $\alpha_{o_k}^{[l_k]}\alpha_{i_k}^{[l_k]}$, where $1 \le k \le N$ is an integer, occurs at $k = r$. That is

$$\max_{1\le k\le N}\left\{\alpha_{o_k}^{[l_k]}\alpha_{i_k}^{[l_k]}\right\} = \alpha_{o_r}^{[l_r]}\alpha_{i_r}^{[l_r]}.$$

Then, the *d.o.c.* of the *delay margin provided by* Corollary 3 at τ_m with aspect ratio vector $v := [v_1 \dots v_N]$, $\min_{1\le k\le N}(v_k) = 1$, satisfies

$$\text{d.o.c.}_{\mathcal{P}}(\tau_m, v) \le \frac{\alpha_{o_r}^{[l_r]}\alpha_{i_r}^{[l_r]} - 1}{\alpha_{o_r}^{[l_r]}\alpha_{i_r}^{[l_r]}}.$$

Furthermore, $\text{d.o.c.}_{\mathcal{P}}(\tau_m, v) < \overline{R}_{o_r}^{[l_r]} + \overline{R}_{i_r}^{[l_r]}$, and $\text{d.o.c.}(\tau_m, v) \to 0$ as $\min_{1\le k\le N}(l_k) \to \infty$.

Proof: The proof follows closely to that of Theorem 5. Let δ^* be the *delay margin* of (1) at $\tau_m = [\tau_{1m}, \dots, \tau_{Nm}]$ with the aspect ratio vector $v = [v_1 \dots v_N]$. Define δ_o^* as the *delay margin provided by* Corollary 3 with $M_o(\theta, s)$ chosen as (7) where each Padé order l_k, $k = 1, \dots, N$ for the kth parameter-dependent filter is chosen sufficiently large $l_k \ge \max(l_{o_k}^{\kappa}, l_{i_k}^{\kappa})$. That is

$$\delta_o^* := \sup\left\{\delta_o | \Sigma_o^m(\delta_o v;\theta) \text{ is asymptotically stable, for all } \theta \in \prod_{k=1}^{N}(0, \delta_o v_k]\right\}.$$

Define δ_i^*

$$\delta_i^* := \sup\left\{\delta_i | \Sigma_i^m(\delta_i v;\theta) \text{ is asymptotically stable, for all } \theta \in \prod_{k=1}^{N}(0, \delta_i v_k]\right\}$$

where the system $\Sigma_i^m(\delta_i v;\theta)$ is formed as in (11) using $M_i(\theta, s)$ with the same Padé orders l_k. Due to the dilation relationship between $h_{i_k}(\theta_k, s)$ and $h_{o_k}(\theta_k, s)$, the following equality holds for at least one integer p, $1 \le p \le N$:

$$\frac{\delta_i^*}{\alpha_{i_p}^{[l_p]}} = \alpha_{o_p}^{[l_p]}\delta_o^*$$

Furthermore, similar to the single-delay case, we have

$$\delta_i^* \ge \delta^* \ge \delta_o^*.$$

Then, a bound on the degree of conservatism follows from:

$$\begin{aligned}\text{d.o.c.}_{\mathcal{P}}(\tau_m) &= \frac{\delta^* - \delta_o^*}{\delta^*} = 1 - \frac{\delta_o^*}{\delta^*}\\ &\le 1 - \frac{\delta_o^*}{\delta_i^*} = \frac{\alpha_{o_p}^{[l_p]}\alpha_{i_p}^{[l_p]} - 1}{\alpha_{o_p}^{[l_p]}\alpha_{i_p}^{[l_p]}} \le \frac{\alpha_{o_r}^{[l_r]}\alpha_{i_r}^{[l_r]} - 1}{\alpha_{o_r}^{[l_r]}\alpha_{i_r}^{[l_r]}}.\end{aligned}$$

The remainder follows in a fashion similar to Theorem 5.

V. Stability Analysis

It has been shown that stability or instability of an uncertain time-delay system can be ascertained by analyzing the stability of a family of finite-dimensional, parameter-dependent systems. Furthermore, this problem reformulation may be done with arbitrarily small conservatism. It is desirable that the new problem of finite-dimensional system analysis be carried out with minimal conservatism introduced. As this problem is one of *real-μ* analysis, it cannot be performed in a lossless manner except for in the single parameter case.

A. Single Delay

In this section, we examine the stability of the comparison system $\Sigma_o(\theta)$. Then, if $\Sigma_o(\theta)$ is stable for all $0 < \theta \le b$, by Corollary 1 the stability of Σ_d is guaranteed. For convenience in notation, throughout the rest of this paper we denote $\rho := (\tau_m - \alpha_o b)^{-1}$ and $\eta := (1/2)\alpha_o^{-1}$.

Let $\left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right]$ be a minimal realization of $p_l(s)I_q$. Also $n_p = ql$ denotes the order of A_p. Then, it is easy to verify that $\left[\begin{array}{c|c} A_o(\theta)_{2n_p\times 2n_p} & B_o(\theta)_{2n_p\times q} \\ \hline C_o(\theta)_{q\times 2n_p} & D_{o\ q\times q} \end{array}\right] :=$ $\left[\begin{array}{c|c} \begin{bmatrix} \rho A_p & \rho\theta^{-(1/2)}B_pC_p \\ 0 & \eta\theta^{-1}A_p \end{bmatrix} & \begin{bmatrix} \rho B_pD_p \\ \eta\theta^{-(1/2)}B_p \end{bmatrix} \\ \hline [C_p \quad \theta^{-(1/2)}D_pC_p] & [D_pD_p] \end{array}\right]$ is a minimal realization of $h_o(\theta, s)I_q = P_l([\tau_m - \alpha_o b]s)P_l(2\alpha_o\theta s)I_q$. This realization may be used to develop the comparison system $\Sigma_o(b,\theta)$ and arrive at the following theorem.

Theorem 8: Let

$$A_L(\theta) = \begin{bmatrix} A_{11} & \theta^{-\frac{1}{2}}A_{12} \\ \theta^{-\frac{1}{2}}A_{21} & \theta^{-1}A_{22} \end{bmatrix} \tag{35}$$

where

$$A_{11} = \begin{bmatrix} A + H_dD_p^2F_d & H_dC_p \\ \rho B_pD_pF_d & \rho A_p \end{bmatrix} \quad A_{12} = \begin{bmatrix} H_dD_pC_p \\ \rho B_pC_p \end{bmatrix}$$
$$A_{21} = [\eta B_pF_d \quad 0] \quad A_{22} = [\eta A_p].$$

Then, the system Σ_d is asymptotically stable for any constant time-delay $\tau \in [\tau_m - b, \tau_m + b]$ if the following two equivalent conditions hold:

$c1$) $A_L(\theta)$ is Hurwitz for all $\theta \in (0, b]$.

$c2$) For every $\theta \in (0, b]$ there exists a symmetric and positive definite matrix $X(\theta) \in \mathbb{R}^{(n+2n_p)\times(n+2n_p)}$ satisfying

$$A_L(\theta)^TX(\theta) + X(\theta)A_L(\theta) < 0.$$

Proof: First, we notice that $c1$) and $c2$) are equivalent. The closed-loop system for the interconnection $\Sigma_o(\theta)$ is given by

$$\begin{bmatrix} \dot{x} \\ \dot{x}_o \end{bmatrix} = \begin{bmatrix} A + H_dD_oF_d & H_dC_o(\theta) \\ B_o(\theta)F_d & A_o(\theta) \end{bmatrix} \begin{bmatrix} x \\ x_o \end{bmatrix}.$$

Therefore, $c1$) and $c2$) hold, if and only if $\Sigma_o(b,\theta)$ is asymptotically stable for all $\theta \in (0, b]$. The conclusion then follows from Corollary 1. ■

To determine stability of the LTDS (1), therefore requires the examination of whether a parameter-dependent matrix is Hurwitz. Two approaches to this analysis problem are presented.

1) LMI Analysis: Theorem 9: Given τ_m, the system Σ_d is asymptotically stable for any constant time-delay $\tau \in [\tau_m - b, \tau_m + b]$, if there exist symmetric matrices $X_2 \in \mathbb{R}^{(n+n_p)\times(n+n_p)}$, $X_3 \in \mathbb{R}^{n_p\times n_p}$, a positive definite matrix $X_1 \in \mathbb{R}^{(n+n_p)\times(n+n_p)}$, a negative–definite matrix $X_4 \in \mathbb{R}^{n_p\times n_p}$ and a matrix $Z \in \mathbb{R}^{(n+n_p)\times n_p}$ such that

$$\Pi(0) < 0 \quad \Pi(b) < 0 \tag{36}$$

and

$$\begin{bmatrix} X_1 + bX_2 & bZ \\ bZ^T & bX_3 + b^2X_4 \end{bmatrix} > 0 \tag{37}$$

where

$$\Pi(\theta) = \begin{bmatrix} \Pi_{11}(\theta) & \Pi_{12}(\theta) \\ * & \Pi_{22}(\theta) \end{bmatrix}$$
$$\Pi_{11}(\theta) = A_{21}^TZ^T + ZA_{21} + (X_1 + \theta X_2)A_{11} + A_{11}^T(X_1 + \theta X_2)$$
$$\Pi_{12}(\theta) = ZA_{22} + \theta A_{11}^TZ + A_{21}^T(X_3 + \theta X_4) + (X_1 + \theta X_2)A_{12}$$
$$\Pi_{22}(\theta) = \theta Z^TA_{12} + \theta A_{12}^TZ + (X_3 + \theta X_4)A_{22} + A_{22}^T(X_3 + \theta X_4).$$

Proof: Note that (36) implies that

$$\Pi(\theta) < 0 \qquad \forall\theta \in (0, b]. \tag{38}$$

Similarly, (37), along with $X_1 > 0$ and $X_4 < 0$ implies that

$$\begin{bmatrix} X_1 + \theta X_2 & \theta Z \\ \theta Z^T & \theta X_3 + \theta^2X_4 \end{bmatrix} > 0 \qquad \forall\theta \in (0, b].$$

Pre- and postmultiplying by $E_\theta = \operatorname{diag}\{I_{n+n_p}, \theta^{-(1/2)}I_{n_p}\}$, the previous inequality is equivalent to

$$X(\theta) := \begin{bmatrix} X_1 + \theta X_2 & \theta^{\frac{1}{2}}Z \\ \theta^{\frac{1}{2}}Z^T & X_3 + \theta X_4 \end{bmatrix} > 0 \qquad \forall\theta \in (0, b].$$

Multiplying (38) on both sides by E_θ yields

$$E_\theta\Pi(\theta)E_\theta = \begin{bmatrix} \Pi_{11} & \theta^{-\frac{1}{2}}\Pi_{12} \\ \theta^{-\frac{1}{2}}\Pi_{12}^T & \theta^{-1}\Pi_{22} \end{bmatrix} < 0$$

which immediately gives

$$A_L(\theta)^TX(\theta) + X(\theta)A_L(\theta) < 0$$

where $A_L(\theta)$ is given by (35). The result follows directly from Theorem 8. ■

1) Eigenvalue Criterion: Herein, an alternative method for the single delay case is provided, which incurs no additional conservatism to that of Theorem 8.

Theorem 10: Given τ_m, $\Sigma_o(b,\theta)$ is asymptotically stable for all $0 < \theta \leq b$, if and only if the two following conditions hold:

$$A_0 \text{ is Hurwitz} \tag{39}$$
$$-(A_0 \oplus A_0)^{-1}(A_1 \oplus A_1) \text{ has no positive real eigenvalue} \tag{40}$$

where

$$A_0 = \begin{bmatrix} A_{11} & \frac{1}{b}A_{12} \\ A_{21} & \frac{1}{b}A_{22} \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

To prove Theorem 10, we need the following lemma.

Lemma 18: [2] Let $\widehat{A}(q) = A_0 + qA_1$, where A_0 and A_1 are constant square matrices. Suppose A_0 is Hurwitz and let

$$\overline{q}^* = \sup\left\{\overline{q} | \widehat{A}(q) \text{ is Hurwitz } \forall q \in [0, \overline{q}]\right\}.$$

Then

$$\overline{q}^* = \frac{1}{\lambda^+_{\max}\left(-(A_0 \oplus A_0)^{-1}(A_1 \oplus A_1)\right)}.$$

Proof: (of Theorem 10): Let $E_\theta = \operatorname{diag}\{\theta^{-(1/2)} I_{n+n_p}, I_{n_p}\}$. Then, $A_L(\theta)$ is Hurwitz if and only if

$$\widehat{A}(\theta) = E_\theta A_L(\theta) E_\theta^{-1} = \begin{bmatrix} A_{11} & \theta^{-1}A_{12} \\ A_{21} & \theta^{-1}A_{22} \end{bmatrix}$$

is Hurwitz. Let $\gamma := \theta^{-1}$. Then, $\widehat{A}(\theta)$ is Hurwitz for all $0 < \theta \le b$ if and only if $\widehat{A}(\gamma) := \begin{bmatrix} A_{11} & \gamma A_{12} \\ A_{21} & \gamma A_{22} \end{bmatrix}$ is Hurwitz for all $(1/b) \le \gamma < \infty$. Let $q = \gamma - (1/b)$. Then, $\widehat{A}(\gamma)$ is Hurwitz for all $(1/b) \le \gamma < \infty$ if and only if

$$\widehat{A}(q) = A_0 + qA_1$$

is Hurwitz for all $0 \le q < \infty$. Note that by definition $\lambda^+_{\max}(M) \to 0^+$ when M does not have any positive real eigenvalues. The conclusion then, follows immediately from Lemma 18. ■

Remark 7: Given τ_m, either Theorems 9 or 10 may be used to confirm stability of $\Sigma_o(b, \theta)$ for a chosen b. The largest value of b may be found through a bisection. Once this largest value of b is found, Corollary 1 then implies stability of Σ_d for all $\tau \in [\tau_m - b, \tau_m + b]$.

B. Multiple-Delay

In this section, we examine the stability of the comparison system $\Sigma_o^m(b, \theta)$. Then, if $\Sigma_o^m(b, \theta)$ is stable for all $\theta \in \prod_{k=1}^N (0, b_k]$, by Corollary 3 the stability of Σ_d on $\prod_{k=1}^N [\tau_{m_k} - b_k, \tau_{m_k} + b_k]$ is guaranteed. Let $\left[\begin{array}{c|c} A_{p_k} & B_{p_k} \\ \hline C_{p_k} & D_{p_k} \end{array}\right]$ be a minimal realization of $P_l(s) I_{q_k}$. Also $n_{p_k} = q_k l_k$ denotes the order of A_{p_k}. Then, a minimal realization of $M_o(\theta, s)$, defined in (7) is given by $\left[\begin{array}{c|c} A_o(\theta) & B_o(\theta) \\ \hline C_o(\theta) & D_o \end{array}\right]$, where

$$\begin{aligned} A_o(\theta) &= \operatorname{diag}\{A_{o_1}(\theta_1) \ldots A_{o_N}(\theta_N)\} \\ B_o(\theta) &= \operatorname{diag}\{B_{o_1}(\theta_1) \ldots B_{o_N}(\theta_N)\} \\ C_o(\theta) &= \operatorname{diag}\{C_{o_1}(\theta_1) \ldots C_{o_N}(\theta_N)\} \\ D_o &= \operatorname{diag}\{D_{o_1} \ldots D_{o_N}\} \end{aligned}$$

where $\left[\begin{array}{c|c} A_{o_k}(\theta_k) & B_{o_k}(\theta_k) \\ \hline C_{o_k}(\theta_k) & D_{o_k} \end{array}\right]$ is a minimal realization of $h_{o_k}(\theta_k, s) I_{q_k}$. For convenience in notation, we define $\rho_k := (\tau_m - \alpha_{o_k} b_k)^{-1}$ and $\eta_k := (1/2)\alpha_{o_k}^{-1}$.

Theorem 11: Let

$$A_L(\theta) := \begin{bmatrix} A_{11} & A_{12} & A_{13}\Gamma^{-\frac{1}{2}} \\ A_{21} & A_{22} & \Gamma^{-\frac{1}{2}} A_{23} \\ \Gamma^{-\frac{1}{2}} A_{31} & 0 & \Gamma^{-1} A_{33} \end{bmatrix} \tag{41}$$

where

$$\begin{aligned} \Gamma &= \operatorname{diag}\left\{\theta_1 I_{n_{p_1}}, \ldots, \theta_N I_{n_{p_N}}\right\} \\ A_{11} &= A + H_d D_o F_d \\ A_{12} &= [H_1 C_{p_1} \; H_2 C_{p_2} \; \ldots \; H_N C_{p_N}] \\ A_{13} &= [H_1 D_{p_1} C_{p_1} \; H_2 D_{p_2} C_{p_2} \; \ldots \; H_N D_{p_N} C_{p_N}] \\ A_{21} &= \left[\rho_1 F_1^T D_{p_1}^T B_{p_1}^T \; \rho_2 F_2^T D_{p_2}^T B_{p_2}^T \; \ldots \; \rho_N F_N^T D_{p_N}^T B_{p_N}^T\right]^T \\ A_{22} &= \operatorname{diag}\{\rho_1 A_{p_1}, \rho_2 A_{p_2}, \ldots, \rho_N A_{pN}\} \\ A_{23} &= \operatorname{diag}\{\rho_1 B_{p_1} C_{p_1}, \rho_2 B_{p_2} C_{p_2}, \ldots, \rho_N B_{p_N} C_{p_N}\} \\ A_{31} &= \left[\eta_1 F_1^T B_{p_1}^T \; \eta_2 F_2^T B_{p_2}^T \; \ldots \; \eta_N F_N^T B_{p_N}^T\right]^T \\ A_{33} &= \operatorname{diag}\{\eta_1 A_{p_1}, \eta_2 A_{p_2}, \ldots, \eta_N A_{p_N}\}. \end{aligned}$$

Then, the system Σ_d is asymptotically stable for any constant time-delay vector $\tau \in \prod_{k=1}^N [\tau_{km} - b_k, \tau_{km} + b_k]$ if the following two equivalent conditions hold.

$c1)$ $A_L(\theta)$ is Hurwitz for all $\theta \in \Theta \equiv \prod_{k=1}^N (0, b_k]$.

$c2)$ For every $\theta \in \Theta \equiv \prod_{k=1}^N (0, b_k]$ there exists a symmetric and positive–definite matrix $X(\theta) \in \mathbb{R}^{(n+2m)\times(n+2m)}$, $m = \sum_{k=1}^N n_{p_k}$, satisfying

$$A_L(\theta)^T X(\theta) + X(\theta) A_L(\theta) < 0.$$

Proof: The proof can be found in [25]. ■

1) LMI Analysis: In this section, we develop a finite set of *linear matrix inequalities* that allow us to verify condition $(c2)$ in Theorem 11. Some additional conservatism may be introduced as a choice of basis functions for $X(\theta)$ is made so as to obtain a finite set.

Theorem 12: Given $\tau_m := [\tau_{1m} \ldots \tau_{Nm}]$, the system Σ_d is asymptotically stable on $\prod_{k=1}^N [\tau_{km} - b_k, \tau_{km} + b_k]$, if there exist symmetric matrices $Y_0 \in \mathbb{R}^{n\times n}$, $Y_k \in \mathbb{R}^{n\times n}$, $k = 1, \ldots, N$ and $X_1 \in \mathbb{R}^{m\times m}$, $X_2 \in \mathbb{R}^{m\times m}$, symmetric block diagonal matrix $X_3 \in \mathbb{R}^{m\times m}$, negative definite block diagonal matrix $X_4 \in \mathbb{R}^{m\times m}$ and matrices $W_1 \in \mathbb{R}^{n\times m}$, $W_2 \in \mathbb{R}^{n\times m}$, $Z \in \mathbb{R}^{m\times m}$, such that for all 2^N vertices b_{v_i}, $i = 1, \ldots, 2^N$ of the polytope $\prod_{k=1}^N [0, b_k]$

$$\Pi(b_{v_i}) < 0, \text{ and } X(b_{v_i}) > 0 \tag{42}$$

where

$$\Pi(\theta)=\begin{bmatrix}\Pi_{11} & \Pi_{12} & \Pi_{13}\\ * & \Pi_{22} & \Pi_{23}\\ * & * & \Pi_{33}\end{bmatrix}$$

$$X(\theta)=\begin{bmatrix}Y(\theta) & W_1 & W_2\Gamma^{\frac{1}{2}}\\ W_1^T & X_1+\Gamma X_2 & Z\Gamma^{\frac{1}{2}}\\ \Gamma^{\frac{1}{2}}W_2^T & \Gamma^{\frac{1}{2}}Z^T & X_3+\Gamma X_4\end{bmatrix}$$

$$\begin{aligned}
Y(\theta) &= Y_0+\sum_{k=1}^{N}\theta_k Y_k\\
\Pi_{11} &= Y(\theta)A_{11}+W_1A_{21}+W_2A_{31}+A_{11}^TY(\theta)\\
&\quad +A_{21}^TW_1^T+A_{31}^TW_2^T\\
\Pi_{12} &= Y(\theta)A_{12}+W_1A_{22}+A_{11}^TW_1+A_{21}^TX_1+A_{21}^TX_2\Gamma\\
&\quad +A_{31}^TZ^T\\
\Pi_{13} &= Y(\theta)A_{13}+W_1A_{23}+W_2A_{33}+A_{11}^TW_2\Gamma\\
&\quad +A_{31}^TX_3+A_{31}^TX_4\Gamma+A_{21}^TZ\Gamma\\
\Pi_{22} &= W_1^TA_{12}+X_1A_{22}+\Gamma X_2A_{22}+A_{12}^TW_1\\
&\quad +A_{22}^TX_1+A_{22}^TX_2\Gamma\\
\Pi_{23} &= W_1^TA_{13}+X_1A_{23}+A_{12}^TW_2\Gamma+\Gamma X_2A_{23}+ZA_{33}\\
&\quad +A_{22}^TZ\Gamma\\
\Pi_{33} &= \Gamma W_2^TA_{13}+X_3A_{33}+\Gamma X_4A_{33}+A_{13}^TW_2\Gamma\\
&\quad +A_{33}^TX_3+A_{33}^TX_4\Gamma+\Gamma Z^TA_{23}+A_{23}^TZ\Gamma.
\end{aligned}$$

Here, X_3 and X_4 are block diagonal matrices with N blocks where the k^{th} block of X_3 and X_4 belong to $\mathbb{R}^{n_{p_k}\times n_{p_k}}$.

Proof: The proof can be found in [25]. ■

Remark 8: Given τ_m and an aspect ratio vector $v := [v_1\dots v_N]$, the largest value of (scalar) b_s for which Theorem 12 affirms stability of $\Sigma_o^m(b,\theta)$ for $b := [b_sv_1\dots b_sv_N]$, may be found through a bisection algorithm. Once this largest value of b_s is found, Corollary 3 implies stability of Σ_d for all $\tau\in\prod_{k=1}^N[\tau_{km}-b_sv_k,\tau_{km}+b_sv_k]$.

VI. Numerical Examples

A. Single Delay

In this section, we examine the stability of some linear time delay systems using presented criteria. The result of [22] is used for comparison in the single-delay case, as it provides an accurate accounting of the stability interval for small problems with a single delay.

Example 1: Consider system (1) with $N=1$ and:

$$A=\begin{bmatrix}-10.9 & -1.9 & -5.2\\ 8.7 & -0.8 & 1.3\\ 11.9 & 3.8 & 5.6\end{bmatrix}\quad A_d=\begin{bmatrix}1.1 & 0.3 & 0.3\\ 0.3 & 1.1 & 1.5\\ 1.6 & 0.4 & 1.1\end{bmatrix}.$$

Note that $A+A_d$ is not Hurwitz and interval stability can not be examined using delay-dependent criteria. Given $\tau_m=0.67$, Table II summarizes the results of the computations. For this example problem, Theorem 10 provided *d.o.c.* results less than the bound (33), as guaranteed by Theorem 5. While the results

TABLE II
COMPUTATIONAL RESULTS FOR EXAMPLE 1

Method	b_{max}	Stability Interval	d.o.c.	bound on d.o.c.
[22]		[0.3130,1.0281]	0	Eq.(33)
Theorem	9			
l=4	0.3209	[0.3491,0.9909]	10.08%	10.14%
l=5	0.3542	[0.3158,1.0242]	0.75%	0.76%
l=6	0.3567	[0.3133,1.0267]	0.056%	0.058%
Theorem	10			
l=4	0.3242	[0.3491,0.9909]	10.08%	10.14%
l=5	0.3542	[0.3158,1.0242]	0.75%	0.76%
l=6	0.3567	[0.3133,1.0267]	0.056%	0.058%

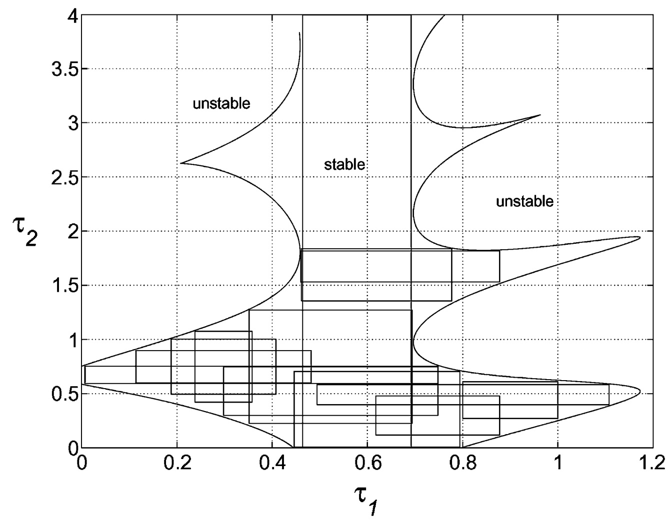

Fig. 2. Stability region in $\tau_1-\tau_2$ plane (enclosed within the curves), and stability regions guaranteed by LMIs (inside the boxes).

provided by Theorem 9 may introduce some conservatism to that of the comparison system, for the problem examined, no additional conservatism is incurred.

B. Two Delays

In this section, we examine the stability of a system with two independent delays. Note that $A+A_1+A_2$ is not Hurwitz.

Example 2: Consider system (1) with $N=2$ and

$$A=\begin{bmatrix}-3.0881 & 2.6698\\ -9.7983 & 2.8318\end{bmatrix}\ \text{and}$$
$$A_1=\begin{bmatrix}0.5645 & 0.0178\\ 1.2597 & 0.8020\end{bmatrix}\quad A_2=\begin{bmatrix}0.4176 & 0.0144\\ 0.9432 & 0.5976\end{bmatrix}.$$

Fig. 2 shows the stability boundary and stable region as provided by careful numerical analysis (by Continuation methods, see [25]). Also shown are 13 stability "boxes," each of which was obtained by using Theorem 12 with a given τ_m and aspect ratio vector v. The area inside each box has been guaranteed stable by the LMI condition. Note that in each of the 13 analyses, the box obtained was the largest that could be obtained with the given aspect ratio. That is, a corner of the box is (nearly) touching a stability boundary.

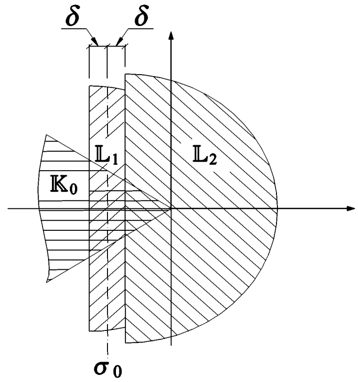

Fig. 3. Scheme of compact regions $\mathbb{L}_1$, $\mathbb{L}_2$, and $\mathbb{K}_0$.

VII. Conclusion

The results in this paper establish a method for converting the problem of analyzing the stability of a family of infinite-dimensional systems to one that considers finite-dimensional systems. The time-delay element is replaced by a parameter-dependent filter in a similar manner to that previously employed for delay-dependent stability [33]. However, the problem treated here is fundamentally different in one important aspect: the delay interval does not include zero and so there is no common member of the time-delay system and comparison system families. As a consequence, comparison system stability does not directly yield LTDS nominal stability, a condition desirable for the establishment of robust stability via value set covering. This difficulty is overcome in this paper by the use of a special homotopy connecting members of the two families such that the value set of the feedback element throughout the continuation remains within that of the parameter-dependent filter, a condition we refer to as strong covering. Thus, robust stability of the LTDS is bootstrapped from the comparison system without establishing (explicitly) nominal stability of any member.

Properties are given for (outer) parameter-dependent filters such that replacement of the delay elements by the filters will produce a sufficient comparison system for stability of the LTDS. Herein, properties are also provided for (inner) parameter-dependent filters that will yield a necessary comparison system, which may be used to establish instability of the LTDS. A constructive method is then provided for generation of both outer and inner filters using Padé approximations such that the desired filter properties are obtained for any delay interval. An *a priori* known bound on the degree-of-conservatism of the sufficient comparison system is obtained. This bound may be reduced to any desired degree by increasing Padé order l and rapidly converges to zero. Furthermore, the bound is independent of all problem data except the ratio $\overline{\tau}/\underline{\tau}$. It is shown that if the LTDS is asymptotically stable over a given closed interval, then there exists a finite-dimensional comparison system that will prove it to be so.

Herein, two methods are explored for stability analysis of the outer (sufficient) comparison system: 1) an eigenvalue technique for the single-delay case which is lossless; and 2) an LMI approach suitable for the multiple-delay problem. An important advantage of our approach is the relative ease with which it may be extended to the problems of $\mathcal{H}_\infty$ analysis and controller synthesis. Some results along these lines have recently been obtained in [25] and [26].

Appendix I
Proof of Lemma 2

First, we need to establish some preliminary results. Define

$$\psi_{\widetilde{do}}(\lambda, s) := \det\Big((sI - A)d_o(\widetilde{\theta}, \lambda s) - A_d e^{-\tau(1-\lambda)s} n_o(\widetilde{\theta}, \lambda s)\Big)$$

$$\sigma_{\widetilde{do}}(\lambda) := \sigma\left(\psi_{\widetilde{do}}(\lambda, s)\right) \quad \lambda \in [0,1]$$

where $h_o(\widetilde{\theta}, s) = n_o(\widetilde{\theta}, s)/d_o(\widetilde{\theta}, s)$. We then have Lemma 19.

Theorem 13 (Rouché's Theorem [27]): If $\Phi(z)$ and $\Psi(z)$ are functions continuous on a closed set $F \subset \mathbb{C}$ and analytic in its interior G, and if $|\Phi(z)| < |\Psi(z)|$ on the boundary of the set F, then the functions $\Psi(z)$ and $\Psi(z) + \Phi(z)$ have the same number of roots in G, counting each root as many times as its multiplicity indicates.

Lemma 19: The function $\sigma_{\widetilde{do}}(\lambda)$ is a finite and continuous function of λ over the interval [0,1].

Proof: We need to show that for all $\lambda_0 \in [0,1]$

$$\forall\delta > 0,\ \exists\varepsilon > 0 \text{ s.t. if } |\lambda_1 - \lambda_0| < \varepsilon \text{ and } \lambda_1 \in [0,1] \text{ then } \left|\sigma_{\widetilde{do}}(\lambda_1) - \sigma_{\widetilde{do}}(\lambda_0)\right| < \delta. \tag{43}$$

It follows from the general theory of functional differential equations that for any fixed $\lambda_0 \in [0,1]$, $\psi_{\widetilde{do}}(\lambda_0, s) = 0$ has a solution in $\mathbb{C}$ and that $\sigma_{\widetilde{do}}(\lambda_0) < \infty$ (e.g., see [13] or [14]). Furthermore, $\exists s_0 \in \mathbb{C}$, such that $\psi_{\widetilde{do}}(\lambda_0, s_0) = 0$ and $\sigma_{\widetilde{do}}(\lambda_0) = \mathrm{Re}(s_0)$. Moreover, any finite region of the complex plane contains at most only a finite number of elements of the set $S_0 := \{s | \psi_{\widetilde{do}}(\lambda_0, s) = 0\}$.

Let us denote $\sigma_0 := \sigma_{\widetilde{do}}(\lambda_0)$. Let $\{s_1, \ldots, s_n\}$ be the n roots of $d_o(\widetilde{\theta}, s) = 0$, and let $\beta := \max_{1\le i\le n} \tan^{-1}((|\mathrm{Im}(s_i)|/|\mathrm{Re}(s_i)|))$. Then, $\beta \in [0, (\pi/2))$ and let $\beta_2 \in (\beta, (\pi/2))$ be an arbitrary real value. Define the set

$$\mathbb{K}_0 := \left\{s \in \mathbb{C} | \mathrm{Re}(s) < 0,\ \frac{|\mathrm{Im}(s)|}{|\mathrm{Re}(s)|} < \tan(\beta_2)\right\}$$

and denote $\mathbb{K} := \mathbb{K}_0 \cup \partial\mathbb{K}_0$. Then, it follows from stability of $d_o(\widetilde{\theta}, s)$ that for all $\lambda \in (0,1]$ all the roots of $d_o(\widetilde{\theta}, \lambda s) = 0$ must lie in $\mathbb{K}_0$. Given δ, define

$$\mathbb{K}_1 := \{s \in \mathbb{C} | s \notin \mathbb{K}_0, \mathrm{Re}(s) \in [\sigma_0 - \delta, \sigma_0 + \delta], \}$$
$$\mathbb{K}_2 := \{s \in \mathbb{C} | s \notin \mathbb{K}_0, \mathrm{Re}(s) \ge \sigma_0 + \delta.\}$$

Also, define closed compact sets

$$\widetilde{\mathbb{K}}_1 := \{s \in \mathbb{K} | \mathrm{Re}(s) \in [\sigma_0 - \delta, \sigma_0 + \delta], \}$$
$$\widetilde{\mathbb{K}}_2 := \{s \in \mathbb{K} | \mathrm{Re}(s) \ge \sigma_0 + \delta.\}$$

By construction, for all $\lambda \in (0,1]$, $d_o(\widetilde{\theta}, \lambda s)$ does not vanish in $\mathbb{K}_1$ and $\mathbb{K}_2$. Moreover, Property $P_O - 3$ implies that $h_o(\widetilde{\theta}, 0) = 1$. Therefore, for each $\lambda \in [0,1]$, $B_1(\lambda) := \sup_{s\in\mathbb{K}_1} (|n_o(\theta, \lambda s)|/|d_o(\theta, \lambda s)|)$ and $B_2(\lambda) := \sup_{s\in\mathbb{K}_2} (|n_o(\theta, \lambda s)|/|d_o(\theta, \lambda s)|)$ exist and are finite. Consider the closed compact set

$$\mathbb{L}_1 := \left\{ \begin{array}{l} s \in \mathbb{C} | \mathrm{Re}(s) \in [\sigma_0 - \delta, \sigma_0 + \delta], \\ |s| \leq 2\|A\| + 2\|A_d\| Q_1 B_1(\lambda_0) \end{array} \right\} \cup \widetilde{\mathbb{K}}_1$$

where $Q_1 = \max(e^{-\tau(1-\lambda_0)(\sigma_0-\delta)}, e^{-\tau(-\lambda_0)(\sigma_0+\delta)})$. By construction, all the solutions of $\psi_{\widetilde{do}}(\lambda_0, s) = 0$ with $\sigma_0 - \delta \leq \mathrm{Re}(s) \leq \sigma_0 + \delta$ must lie in $\mathbb{L}_1$. Moreover, we can safely assume that δ is sufficiently small so that $\mathbb{L}_1$ contains only those elements of S_0 with $\mathrm{Re}(s) = \sigma_0$. We search for an ε that validates (43). We will limit the search to $\varepsilon \in \mathbb{I} := (0, \min(\lambda_0, 1 - \lambda_0))$. Now, define a closed compact set

$$\mathbb{L}_2 := \left\{ \begin{array}{c} s \in \mathbb{C} | \mathrm{Re}(s) \geq \sigma_0 + \delta, \\ |s| \leq 2\|A\| + 2\|A_d\| Q_2 \overline{B_2} \end{array} \right\} \cup \widetilde{\mathbb{K}}_2$$

where $Q_2 = \max(1, e^{-\tau(\sigma_0+\delta)})$ and $\overline{B_2} = \sup_{\lambda\in[0,1]} B_2(\lambda)$.

For each $s \in \partial\mathbb{L}_1$, define $\delta_{1s} := |\psi_{\widetilde{do}}(\lambda_0, s)|$. Similarly, for each $s \in \partial\mathbb{L}_2$, define $\delta_{2s} := |\psi_{\widetilde{do}}(\lambda_0, s)|$. (Note that by construction $\psi_{\widetilde{do}}(\lambda_0, s)$ is nonvanishing on $\partial\mathbb{L}_1$ and $\partial\mathbb{L}_2$ and, hence, $\delta_{1s} > 0, \forall s \in \partial\mathbb{L}_1$ and $\delta_{2s} > 0, \forall s \in \partial\mathbb{L}_2$). By continuity of the determinant at each $s \in \mathbb{C}$ and, particularly, for each $s \in \partial\mathbb{L}_1$ we have

$$\forall s \in \partial\mathbb{L}_1, \text{ for } \delta_{1s} > 0, \exists \varepsilon_{1s} \in \mathbb{I} \text{ s.t. if } |\lambda_1 - \lambda_0| < \varepsilon_{1s} \text{ then } \left|\psi_{\widetilde{do}}(\lambda_1, s) - \psi_{\widetilde{do}}(\lambda_0, s)\right| < \delta_{1s}.$$

Similarly

$$\forall s \in \partial\mathbb{L}_2, \text{ for } \delta_{2s} > 0, \exists \varepsilon_{2s} \in \mathbb{I} \text{ s.t. if } |\lambda_1 - \lambda_0| < \varepsilon_{2s} \text{ then } \left|\psi_{\widetilde{do}}(\lambda_1, s) - \psi_{\widetilde{do}}(\lambda_0, s)\right| < \delta_{2s}.$$

Define $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ where $\varepsilon_1 := \min_{s\in\partial\mathbb{L}_1} \{\varepsilon_{1s}\}$, and $\varepsilon_2 := \min_{s\in\partial\mathbb{L}_2} \{\varepsilon_{2s}\}$. Clearly, $\varepsilon > 0$ and for all λ_1 with $|\lambda_1 - \lambda_0| < \varepsilon$ the following two inequalities must hold:

$$\left|\psi_{\widetilde{do}}(\lambda_1, s) - \psi_{\widetilde{do}}(\lambda_0, s)\right| < \left|\psi_{\widetilde{do}}(\lambda_0, s)\right|, \ \forall s \in \partial\mathbb{L}_1 \quad (44)$$
$$\left|\psi_{\widetilde{do}}(\lambda_1, s) - \psi_{\widetilde{do}}(\lambda_0, s)\right| < \left|\psi_{\widetilde{do}}(\lambda_0, s)\right|, \ \forall s \in \partial\mathbb{L}_2. \quad (45)$$

Since for all $\lambda \in [0,1]$, $\psi_{\widetilde{do}}(\lambda, s)$ is analytic over $\mathbb{C}$ and particularly over $\mathbb{L}_1$, from Rouché's Theorem and (44), we conclude that $\psi_{\widetilde{do}}(\lambda_0, s)$ and $\psi_{\widetilde{do}}(\lambda_1, s)$ have the same number of roots in $\mathbb{L}_1$ when $|\lambda_1 - \lambda_0| < \varepsilon$. Therefore, $\psi_{\widetilde{do}}(\lambda_1, s)$ has at least one root in $\mathbb{L}_1$. We now, show that $\psi_{\widetilde{do}}(\lambda_1, s)$ cannot have any root outside of $\mathbb{L}_1$ with $\mathrm{Re}(s) \geq \sigma_0 + \delta$. Clearly, $\psi_{\widetilde{do}}(\lambda_1, s)$ cannot have a root with $\mathrm{Re}(s) = \sigma_0 + \delta$ (or either (44) or (45) is violated). Assume that for some $\lambda_1 \in [0,1]$, with $|\lambda_1 - \lambda_0| < \varepsilon$, $\psi_{\widetilde{do}}(\lambda_1, s)$ has a root with $\mathrm{Re}(s) > \sigma_0 + \delta$. By construction this root must lie in $\mathbb{L}_2$. However, (45) is true on the boundary of $\mathbb{L}_2$ and Rouché's Theorem would imply that $\psi_{\widetilde{do}}(\lambda_0, s)$ must have a root in $\mathbb{L}_2$. This contradiction proves that $|\sigma_{\widetilde{do}}(\lambda_1) - \sigma_{\widetilde{do}}(\lambda_0)| < \delta$, completing the proof. ■

Lemma: For all $\lambda \in [0,1]$, we have

$$\begin{aligned} &\text{i)} \sigma_{do}(\lambda) < 0 \Longleftrightarrow \sigma_{\widetilde{do}}(\lambda) < 0 \\ &\text{ii)} \sigma_{do}(\lambda) > 0 \Longleftrightarrow \sigma_{\widetilde{do}}(\lambda) > 0 \\ &\text{iii)} \sigma_{do}(\lambda) = 0 \Longleftrightarrow \sigma_{\widetilde{do}}(\lambda) = 0. \end{aligned}$$

Proof: For $\lambda = 0$, $\psi_{do}(\lambda, s) = \psi_{\widetilde{do}}(\lambda, s)$ and $\sigma_{do}(\lambda) = \sigma_{\widetilde{do}}(\lambda)$ and therefore i)–iii) follow. For $\lambda \in (0,1]$, We first prove that if $\sigma_{do}(\lambda) < 0$ then $\sigma_{\widetilde{do}}(\lambda) < 0$. Assume $\sigma_{do}(\lambda) < 0$ but $\sigma_{\widetilde{do}}(\lambda) \geq 0$. Then, there exists an $s_0 \in \mathbb{C}$ with $\mathrm{Re}(s_0) \geq 0$ such that $\psi_{\widetilde{do}}(\lambda, s_0) = 0$. Since $\mathrm{Re}(s_0) \geq 0$, and the denominator of $h_o(\theta, s)$ is Hurwitz, it follows that $d_o(\widetilde{\theta}, \lambda s_0) \neq 0$. Dividing $\psi_{\widetilde{do}}(\lambda, s_0)$ by $d_o(\widetilde{\theta}, \lambda s_0)$ results in $\psi_{do}(\lambda, s_0) = 0$ and therefore, $\sigma_{do}(\lambda) \geq 0$, contradicting $\sigma_{do}(\lambda) < 0$. Now, assume that $\sigma_{\widetilde{do}}(\lambda) < 0$ but $\sigma_{do}(\lambda) \geq 0$. Then, there exists an $s_0 \in \mathbb{C}$ with $\mathrm{Re}(s_0) \geq 0$ such that $\psi_{do}(\lambda, s_0) = 0$ and $d_o(\widetilde{\theta}, \lambda s_0) \neq 0$. It follows then immediately that $\psi_{\widetilde{do}}(\lambda, s_0) = 0$ and $\sigma_{\widetilde{do}}(\lambda) \geq 0$. This contradiction completes the proof of i). ii) is proved similarly and iii) is an immediate result of i) and ii).

This brings us to the proof of Lemma 2.

Proof: (of Lemma 2): Note that $\psi_{do}(0, s) = \psi_d(\tau, s)$, and $\psi_{do}(1, s) = \psi_o(\widetilde{\theta}, s)$. Therefore, $\sigma_d(\tau) > 0$ and $\sigma_o(\widetilde{\theta}) < 0$ imply that $\sigma_{do}(0) > 0$ and $\sigma_{do}(1) < 0$, respectively. From Lemma 20, we conclude that $\sigma_{\widetilde{do}}(0) > 0$ and $\sigma_{\widetilde{do}}(1) < 0$. From Lemma 19, $\sigma_{\widetilde{do}}(\lambda)$ is continuous for all $\lambda \in [0,1]$ and, therefore, there exists a $\lambda^* \in (0,1)$ such that $\sigma_{\widetilde{do}}(\lambda^*) = 0$. From this (and Lemma 20), $\sigma_{do}(\lambda^*) = 0$ follows directly. From standard results in quasi-polynomials, $\sigma_{do}(\lambda^*) = 0$ implies the existence of $s^* = j\omega^*$ such that $\psi_{do}(\lambda^*, s^*) = 0$. Since the roots are complex conjugates, we only need to consider the non-negative imaginary axis, and the proof is complete. ■

APPENDIX II
PROOFS OF AUXILIARY LEMMAS

A. *Proof of Lemma 7*

First, we prove that there exists a constant $L > 0$, such that

$$\forall \omega \in \mathbb{R} - \{0\} \quad \left|\frac{d}{d\omega} \mathrm{Arg}\,(p_l(j\omega))\right| \leq \frac{L}{\omega^2}. \qquad (46)$$

Using (13), (46) is equivalent to

$$\forall \omega \in \mathbb{R} - \{0\} \quad \frac{T_l(\omega)}{\omega^{2l} + T_l(\omega)} < \frac{L}{\omega^2}.$$

Therefore, it suffices to find a constant $L > 0$, such that

$$\begin{aligned}\sum_{k=0}^{l-1} a_{(l,k)}\omega^{2k+2} &< L\sum_{k=0}^{l-1} a_{(l,k)}\omega^{2k} + L\omega^{2l} \qquad (47)\\ &= L\left(a_{(l,0)} + \sum_{k=0}^{l-1} a_{(l,k+1)}\omega^{2k+2}\right)\\ &< L\sum_{k=0}^{l-1} a_{(l,k+1)}\omega^{2k+2}. \qquad (48)\end{aligned}$$

(To derive (48) note that $a_{(l,l)} = 1$). Clearly, with

$$L := \max_{0\le k\le l-1}\left\{\frac{a_{(l,k)}}{a_{(l,k+1)}}\right\}$$

(47) and therefore (46) holds. Moreover

$$\begin{aligned}\frac{a_{(l,k)}}{a_{(l,k+1)}} &= 2(2l-2k-1)(2l-k)\frac{(k+1)}{(l-k)}\\ &< 4(2l-k)(k+1) < 4(l^2+1).\end{aligned}$$

Therefore, with $L = 4(l^2+1)$, (46) holds. The desired result follows immediately from (46) and (17). Proof is complete.

B. Outer Parameter-Dependent Filter: Proof of Existence and Satisfaction of Properties

Proof: (of Lemma 10): First, consider $\Psi_o(1)$

$$\Psi_o(1) = \mathrm{Arg}\left(p_l\left([\kappa-1]j\omega_o\right)\right) - 2\pi + [\kappa+1]\omega_o.$$

Note that there exists a $\beta_1 > 0$ such that $\mathrm{Arg}(p_l([\kappa-1]j\omega_o)) = \mathrm{Arg}(e^{-[\kappa-1]j\omega_o}) + \beta_1$ where $\beta_1 > 0$ inequality results from inequality (18). Therefore

$$\begin{aligned}\Psi_o(1) &= -[\kappa-1]\omega_o + \beta_1 - 2\pi + [\kappa+1]\omega_o\\ &= 2(\omega_o - \pi) + \beta_1 > 0\end{aligned}$$

where the inequality follows from $\beta_1 > 0$ and $\omega_o > \pi$. Denote $\mu := \max(1, \kappa/1.8)$. Then:

$$\begin{aligned}&\text{i)}\, 1 < \frac{\kappa}{\mu} \le 1.8 < 1 + \frac{\pi}{\omega_o^{[l]}} \qquad \forall l \ge 3\\ &\text{ii)}\, \frac{\kappa}{\mu} \le \kappa.\end{aligned}$$

Note that

$$\begin{aligned}\Psi_o\left(\frac{\kappa}{\mu}\right) = &\mathrm{Arg}\left(p_l\left(\left(\kappa - \frac{\kappa}{\mu}\right)j\omega_o\right)\right)\\ &+\mathrm{Arg}\left(p_l\left(2\frac{\kappa}{\mu}j\omega_o\right)\right) + [\kappa+1]\omega_o.\end{aligned}$$

Furthermore, there exist $\beta_2 > 0$ and $\beta_3 > 0$ such that

$$\begin{aligned}\mathrm{Arg}\left(p_l\left(2\frac{\kappa}{\mu}j\omega_o\right)\right) &= -\frac{2\kappa}{\mu}\omega_o + \beta_2,\ \beta_2 > 0\\ \mathrm{Arg}\left(p_l\left(\left(\kappa - \frac{\kappa}{\mu}\right)j\omega_o\right)\right) &= -\left(\kappa - \frac{\kappa}{\mu}\right)\omega_o + \beta_3,\ \beta_3 > 0.\end{aligned}$$

Then

$$\begin{aligned}\Psi_o\left(\frac{\kappa}{\mu}\right) &= \omega_o\left(1 - \frac{\kappa}{\mu}\right) + \beta_2 + \beta_3\\ &\le \omega_o\left(1 - \frac{\kappa}{\mu}\right) + 2\max(\beta_2, \beta_3).\end{aligned}$$

From Lemma (8), we have

$$\begin{aligned}\left|p_l\left(2\frac{\kappa}{\mu}j\omega_o\right) - e^{-2\frac{\kappa}{\mu}j\omega_o}\right| &< 2\left(\frac{2\kappa e\omega_o}{4l+2}\frac{1}{\mu}\right)^{2l+1}\\ \left|p_l\left(\left(\kappa - \frac{\kappa}{\mu}\right)j\omega_o\right) - e^{-\left(\kappa - \frac{\kappa}{\mu}\right)j\omega_o}\right| &< 2\left(\frac{2\kappa e\omega_o}{4l+2}\frac{\mu-1}{2\mu}\right)^{2l+1}.\end{aligned}$$

Therefore, if

$$2\arcsin\left[\left(\frac{\kappa e\omega_o}{2l+1}\gamma\right)^{2l+1}\right] < \frac{1}{2}\omega_o\left(\frac{\kappa}{\mu} - 1\right)$$

where

$$\gamma := \max\left(\frac{1}{\mu}, \frac{\mu-1}{2\mu}\right)$$

then $2\max(\beta_2, \beta_3) < \omega_o((\kappa/\mu) - 1)$ and $\Psi_o((\kappa/\mu)) < 0$. Since the right-hand side of (25) is bounded from below by $(1/2)\pi((\kappa/\mu) - 1)$, and the left hand side can be made arbitrary small by increasing the Padé order, for a high enough Padé order, the inequality can always be satisfied. Since $\Psi_o(\alpha)$ is a continuous function of α, $\Psi_o((\kappa/\mu)) < 0$ and $\Psi_o(1) > 0$ imply that there exists at least one $\alpha^* \in (1, (\kappa/\mu))$ such that $\Psi_o(\alpha^*) = 0$. From $(d^2/d\alpha^2)\Psi_o(\alpha) > 0$, $\min\{\alpha^* | 1 < \alpha^* < \kappa, \Psi_o(\alpha^*) = 0\}$ exists which we label as α_o. We will now refer to the value of l chosen above as l_o^κ. Since (25) is automatically satisfied for any value of $l \ge l_o^\kappa$, the existence of $\alpha_o \in (1, \kappa)$ for every $l \ge l_o^\kappa$ is assured. ■

Proof: (of Lemma 11): From Lemma 6, $|p_l(j\omega)| = 1$, $\forall\omega \in \mathbb{R}$. Therefore, $|h_o(\theta, j\omega)| = |e^{-j\tau\omega}|$ and only the *argument* need to be considered in proving the result. First, consider the case where $0 \le \omega < \omega_c := \omega_o/(\alpha_o b)$. We only need to demonstrate that

$$\begin{aligned}&\mathrm{Arg}\left(h_o(b, j\omega)\right) \le \mathrm{Arg}(e^{-\overline{\tau}j\omega}) <\\ &\quad\mathrm{Arg}(e^{-\underline{\tau}j\omega}) \le \mathrm{Arg}\left(h_o(0, j\omega)\right),\ \omega \in [0, \omega_c). \qquad (49)\end{aligned}$$

Since $h_o(0,s) = p_l([\tau_m - \alpha_o b]s)$, from inequality (17) we have

$$\frac{d}{d\omega}\mathrm{Arg}\,(h_o(0,j\omega)) \geq -[\tau_m - \alpha_o b] > -[\tau_m - b] = -\underline{\tau} \qquad \forall \omega \geq 0.$$

Then, the rightmost inequality of (49) follows directly since $(d/d\omega)\arg(e^{-\underline{\tau}j\omega}) = -\underline{\tau}$. Now, consider the leftmost inequality of (49). Since $\mathrm{Arg}(p_l(j\omega))$ is convex for $0 < \omega$, $\mathrm{Arg}(h_o(b,j\omega))= \mathrm{Arg}(p_l([\tau_m - \alpha_o b]j\omega)) + \mathrm{Arg}(p_l(2\alpha_o bj\omega))$ is also a convex function of ω. Let $\phi_a(\omega) := \mathrm{Arg}(h_o(b,j\omega)) - \mathrm{Arg}(e^{-j\overline{\tau}\omega}) \equiv \mathrm{Arg}(h_o(b,j\omega)) + \overline{\tau}\omega$. Since $\phi_a(0) = \phi_a(\alpha_o\omega_c) = 0$ and $(d\phi_a(\omega)/d\omega)|_{\omega=0} = -[\tau_m + \alpha_o b] + \overline{\tau} < 0$, convexity of $\phi_a(\omega)$ yields $\phi_a(\omega) \leq 0, \omega \in [0, \alpha_o\omega_c]$. Since $[0,\omega_c) \subset [0, \alpha_o\omega_c]$ the leftmost inequality of (49) is satisfied for $0 \leq \omega < \omega_c$. Now, consider the case where $\omega \geq \omega_c$. Then, $(h_o(\theta, j\omega))$ takes all the values on the unit circle for $0 < \theta \leq b$. Therefore, for every $\tau, \exists\theta \in (0,b]$, such that $e^{-\tau j\omega} = h_o(\theta, j\omega)$. ∎

Proof: (of Lemma 12): For $\omega \geq (\pi/b)$, $\Omega_d(\omega)$ contains all points on the unit circle, so $h_o(\widetilde{\theta}, j\omega) \in \Omega_d(\omega)$ for any $\widetilde{\theta}$ in this case. For $\omega < (\pi/b)$, the desired result is equivalent to

$$-(\kappa+1)\upsilon \leq \Phi_o(\widetilde{\theta},\upsilon) \leq -(\kappa-1)\upsilon \qquad \forall \upsilon < \pi \tag{50}$$

where

$$\Phi_o(\theta,\upsilon) = \mathrm{Arg}\left(p_l\left([\kappa-\alpha_o]j\upsilon\right)p_l\left(2\alpha_o\frac{\theta}{b}j\upsilon\right)\right).$$

Choose $\widetilde{\theta} = (b/\alpha_o)$. Then, $(d/d\upsilon)\Phi_o(\widetilde{\theta},\upsilon)|_{\upsilon=0} = -(\kappa-\alpha_o) - 2 = -(\kappa+1) + (\alpha_o - 1) > -(\kappa+1)$. Since $(d^2/d\upsilon^2)\Phi_o(\widetilde{\theta},\upsilon) > 0$, the left inequality in (50) is satisfied. Furthermore, to establish the right inequality, it suffices to demonstrate that $\Phi_o(\widetilde{\theta},\omega_o) \leq -(\kappa-1)\omega_o$ since $\Phi_o(\widetilde{\theta},0) = 0$, $(d/d\upsilon)\Phi_o(\widetilde{\theta},\upsilon)|_{\upsilon=0} < 0$, and $(d^2/d\upsilon^2)\Phi_o(\widetilde{\theta},\upsilon) > 0$. First, notice that

$$\mathrm{Arg}\left(p_l\left(2\alpha_o\frac{\widetilde{\theta}}{b}j\omega_o\right)\right) = \mathrm{Arg}\left(p_l\left(2j\omega_o\right)\right) = -2\pi.$$

From

$$\mathrm{Arg}\left(p_l\left(j[\kappa-\alpha_o]\omega_o\right)p_l(j2\alpha_o\omega_o)\right) = -(\kappa+1)\omega_o$$

we have

$$\begin{aligned}\Phi_o(\widetilde{\theta},\omega_o) &= -(\kappa+1)\omega_o - 2\pi - \mathrm{Arg}\left(p_l(j2\alpha_o\omega_o)\right)\\ \Phi_o(\widetilde{\theta},\omega_o) &< -(\kappa-1)\omega_o + 2\left(\alpha_o - 1 - \frac{\pi}{\omega_o}\right)\omega_o.\end{aligned}$$

Since for $l \geq l_o^\kappa$, $\alpha_o^{[l]} < 1 + (\pi/\omega_o^{[l]})$, we have $\Phi_o(\widetilde{\theta},\omega_o) < -(\kappa-1)\omega_o$, which guarantees the desired result. ∎

Proof: (of Lemma 13): Define $\upsilon := (\kappa - \alpha_o^{[l]})\omega_o^{[l]}$, $w := 2\alpha_o^{[l]}\omega_o^{[l]}$, $R_1 := \{1 - e^{j\upsilon}p_l(j\upsilon)\}$, and $R_2 := \{1 - e^{jw}p_l(jw)\}$. It is straightforward then to verify that $R_1R_2 - R_1 - R_2 = e^{j[\upsilon+w]}p_l(j\upsilon)p_l(jw) - 1$. From (21) and (22), $p_l(j\upsilon)p_l(jw) = e^{-j[\kappa+1]\omega_o^{[l]}}$ and, therefore, $|e^{j[\upsilon+w]}p_l(j\upsilon)p_l(jw) - 1| = |e^{-j(1-\alpha_o^{[l]})\omega_o^{[l]}} - 1|$, yielding

$$\left|e^{-j\left(1-\alpha_o^{[l]}\right)\omega_o^{[l]}} - 1\right| \leq |R_1||R_2| + |R_1| + |R_2|$$

or, alternatively

$$2\sin\left(\left[\alpha_o^{[l]} - 1\right]\frac{\omega_o^{[l]}}{2}\right) \leq |R_1||R_2| + |R_1| + |R_2|. \tag{51}$$

Also, note that $|R_1| = |e^{-j\upsilon} - p(j\upsilon)|$ and $|R_2| = |e^{-jw} - p(jw)|$. Then, from Lemma 8, we have

$$\begin{aligned}|R_1| &\leq 2\left(\frac{\upsilon e}{4l+2}\right)^{2l+1} < 2\left(\frac{(\kappa-1)\pi e}{2l+1}\right)^{2l+1} = 2\overline{R}_1^{[l]}\\ |R_2| &\leq 2\left(\frac{we}{4l+2}\right)^{2l+1} < 2\left(\frac{2\kappa\pi e}{2l+1}\right)^{2l+1} = 2\overline{R}_2^{[l]}.\end{aligned}$$

Clearly

$$\lim_{l\to\infty}\overline{R}_1^{[l]} = \lim_{l\to\infty}\overline{R}_2^{[l]} = 0.$$

Furthermore, for $l \geq l_o^\kappa$, we have

$$1 < \alpha_o^{[l]} < 1 + \frac{\pi}{\omega_o^{[l]}}.$$

It then follows that $[\alpha_o^{[l]} - 1](\omega_o^{[l]}/2) \in (0, (\pi/2))$. From the inequality $2y < \pi\sin(y), \forall y \in (0,(\pi/2))$ we have

$$\frac{4}{\pi}\left[\alpha_o^{[l]} - 1\right]\frac{\omega_o^{[l]}}{2} < 2\sin\left(\left[\alpha_o^{[l]} - 1\right]\frac{\omega_o^{[l]}}{2}\right)$$

Now, employing (15) and simplifying the previous inequality, we have

$$\left[\alpha_o^{[l]} - 1\right]\frac{2\omega_o^{[l]}}{\pi} < |R_1||R_2| + |R_1| + |R_2|.$$

Then, from $\pi < \omega_o^{[l]}$ we have $[\alpha_o^{[l]} - 1] < 2\overline{R}_1^{[l]}\overline{R}_2^{[l]} + \overline{R}_1^{[l]} + \overline{R}_2^{[l]}$. Clearly

$$\lim_{l\to\infty}\alpha_o^{[l]} = 1.$$

∎

C. *Proof of Lemma 17*

First notice that

$$\begin{aligned}\operatorname{Arg}\{h_o(\theta,j\omega)\}&\equiv\operatorname{Arg}\{p_l([\tau_m-\alpha_o b]j\omega)\,p_l(2\alpha_o\theta j\omega)\}\\ \operatorname{Arg}\{h_i(\theta,j\omega)\}&\equiv\operatorname{Arg}\left\{p_l\left(\left[\tau_m-\frac{1}{\alpha_i}b\right]j\omega\right)p_l\left(\frac{2}{\alpha_i}\theta j\omega\right)\right\}.\end{aligned}$$

Since $|h_o(\theta,j\omega)|=1$ and $|h_i(\theta,j\omega)|=1$, we have

$$\begin{aligned}&\|h_o(\theta,s)-h_i(\theta,s)\|_\infty\\ &\quad=\sup_{\omega\in\mathbb{R}}\left|2\sin\left(\frac{\operatorname{Arg}\{h_o(\theta,j\omega)\}-\operatorname{Arg}\{h_i(\theta,j\omega)\}}{2}\right)\right|\\ &\quad\le\underbrace{2sup_{\omega\in\mathbb{R}}\left|\int_{(\tau_m-\alpha_o b)\omega}^{\left(\tau_m-\frac{1}{\alpha_i}b\right)\omega}\left|\frac{d}{dv}\operatorname{Arg}\{p_l(jv)\}\right|dv\right|}_{I_1}\\ &\qquad+\underbrace{2\sup_{\omega\in\mathbb{R}}\left|\int_{\frac{2}{\alpha_i}\theta\omega}^{2\alpha_o\theta\omega}\left|\frac{d}{dv}\operatorname{Arg}\{p_l(jv)\}\right|dv\right|}_{I_2}.\end{aligned}$$

Using (19), an upper bound for both I_1 and I_2 can be found in the following way:

$$I_1\le2\sup_{\omega\in\mathbb{R}}\left|\int_{(\tau_m-\alpha_o b)\omega}^{\left(\tau_m-\frac{1}{\alpha_i}b\right)\omega}\min\left(1,\frac{L}{v^2}\right)dv\right|.$$

Now, consider the two following cases:

$$\text{i)}\quad|\omega|\ge\frac{\sqrt{L}}{(\tau_m-\alpha_o b)},\qquad\text{ii)}\quad\frac{\sqrt{L}}{(\tau_m-\alpha_o b)}>|\omega|.$$

Case i):

$$\begin{aligned}I_1&\le2\sup_{\omega\in\mathbb{R}}\left|\int_{(\tau_m-\alpha_o b)\omega}^{\left(\tau_m-\frac{1}{\alpha_i}b\right)\omega}\frac{L}{v^2}dv\right|\\ &=2\sup\frac{bL}{|\omega|}\frac{\left(\alpha_o^{[l]}-\frac{1}{\alpha_i^{[l]}}\right)}{\left(\tau_m-\frac{1}{\alpha_i^{[l]}}b\right)\left(\tau_m-\alpha_o^{[l]}b\right)}.\end{aligned}$$

Since $|\omega|(\tau_m-\alpha_o b)\ge\sqrt{L}$ we have

$$\begin{aligned}I_1&\le\frac{2b\sqrt{L}}{\left(\tau_m-\frac{1}{\alpha_i^{[l]}}b\right)}\left(\alpha_o^{[l]}-\frac{1}{\alpha_i^{[l]}}\right)\\ &\le\frac{2b\sqrt{L}}{\left(\tau_m-\frac{1}{\alpha_i^{[l]}}b\right)}\left(\overline{R}_o^{[l]}+\overline{R}_i^{[l]}\right)\end{aligned}\tag{52}$$

where (52) follows from (26) and (31).

Case ii):

$$\begin{aligned}I_1&\le2\sup_{\omega}\left|\int_{(\tau_m-\alpha_o b)\omega}^{\left(\tau_m-\frac{1}{\alpha_i}b\right)\omega}1dv\right|=2\sup_{\omega}|\omega|b\left(\alpha_o^{[l]}-\frac{1}{\alpha_i^{[l]}}\right)\\ &\le\frac{2b\sqrt{L}}{\left(\tau_m-\alpha_o^{[l]}b\right)}\left(\alpha_o^{[l]}-\frac{1}{\alpha_i^{[l]}}\right)\le\frac{2b\sqrt{L}\left(\overline{R}_o^{[l]}+\overline{R}_i^{[l]}\right)}{\left(\tau_m-\alpha_o^{[l]}b\right)}.\end{aligned}$$

Therefore, an upper bound on I_1, for all $\omega\in\mathbb{R}$, is given by

$$I_1\le\frac{2b\sqrt{4(l^2+1)}}{\left(\tau_m-\alpha_o^{[l]}b\right)}\left(\overline{R}_o^{[l]}+\overline{R}_i^{[l]}\right)\le\frac{4b(l+1)\left(\overline{R}_o^{[l]}+\overline{R}_i^{[l]}\right)}{\left(\tau_m-\alpha_o^{[l]}b\right)}.$$

We now, turn our attention to I_2

$$I_2\le2\sup_{\omega\in\mathbb{R}}\left|\int_{\frac{2}{\alpha_i}\theta\omega}^{2\alpha_o\theta\omega}\min\left(1,\frac{L}{v^2}\right)dv\right|.$$

Consider the two following cases:

$$\text{i)}\quad|\omega|\ge\frac{\alpha_i\sqrt{L}}{2\theta}\qquad\text{ii)}\quad\frac{\alpha_i\sqrt{L}}{2\theta}>|\omega|.$$

Case i):

$$I_2\le2\sup_{\omega\in\mathbb{R}}\left|\int_{\frac{2}{\alpha_i}\theta\omega}^{2\alpha_o\theta\omega}\frac{L}{v^2}dv\right|=2\sup\frac{L}{2\theta|\omega|}\left(\alpha_i^{[l]}-\frac{1}{\alpha_o^{[l]}}\right).$$

Since $2\theta|\omega|\ge\alpha_i\sqrt{L}$, we have

$$I_2\le2\frac{\sqrt{L}}{\alpha_i}\left(\alpha_i^{[l]}-\frac{1}{\alpha_o^{[l]}}\right)\le2\sqrt{L}\left(\overline{R}_o^{[l]}+\overline{R}_i^{[l]}\right).$$

Cases ii):

$$\begin{aligned}I_2&\le2\sup_{\omega\in\mathbb{R}}\left|\int_{\frac{2}{\alpha_i}\theta\omega}^{2\alpha_o\theta\omega}1dv\right|=4\theta|\omega|\left(\alpha_o-\frac{1}{\alpha_i}\right)\\ &\le2\alpha_i\sqrt{L}\left(\alpha_o-\frac{1}{\alpha_i}\right)\le2\alpha_i\sqrt{L}\left(\overline{R}_o^{[l]}+\overline{R}_i^{[l]}\right).\end{aligned}$$

Therefore, an upper bound on I_2, for all $\omega\in\mathbb{R}$, is given by

$$I_2\le2\alpha_i\sqrt{4(l^2+1)}\left(\overline{R}_o^{[l]}+\overline{R}_i^{[l]}\right)\le4\alpha_i(l+1)\left(\overline{R}_o^{[l]}+\overline{R}_i^{[l]}\right).$$

We have

$$\begin{aligned}\|h_o(\theta,s)-h_i(\theta,s)\|_\infty &\le I_1+I_2\\ &\le b\left(\alpha_i+\frac{1}{\left(\kappa-\alpha_o^{[l]}\right)}\right)(l+1)\\ &\quad\times\left(\overline{R}_o^{[l]}+\overline{R}_i^{[l]}\right).\end{aligned}$$

It is clear that the right hand side of the above inequality goes to zero as l goes to infinity. Therefore,

$$\lim_{l\to\infty} I_1+I_2=0, \Rightarrow \lim_{l\to\infty}\|h_o(\theta,s)-h_i(\theta,s)\|_\infty=0.$$

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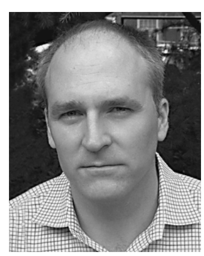

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